

BILATERAL CANONICAL CASCADES: MULTIPLICATIVE REFINEMENT PATHS TO WIENER'S AND VARIANT FRACTIONAL BROWNIAN LIMITS

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ABSTRACT. The original density is 1 for $t \in (0, 1)$, b is an integer base ($b \geq 2$), and $p \in (0, 1)$ is a parameter. The first construction stage divides the unit interval into b subintervals and multiplies the density in each subinterval by either 1 or -1 with the respective frequencies of $\frac{1}{2} + \frac{p}{2}$ and $\frac{1}{2} - \frac{p}{2}$. It is shown that the resulting density can be renormalized so that, as $n \rightarrow \infty$ (n being the number of iterations) the signed measure converges in some sense to a non-degenerate limit. If $H = 1 + \log_b p > 1/2$, hence $p > b^{-1/2}$, renormalization creates a martingale, the convergence is strong, and the limit shares the Hölder and Hausdorff properties of the fractional Brownian motion of exponent H . If $H \leq 1/2$, hence $p \leq b^{-1/2}$, this martingale does not converge. However, a different normalization can be applied, for $H \leq \frac{1}{2}$ to the martingale itself and for $H > \frac{1}{2}$ to the discrepancy between the limit and a finite approximation. In all cases the resulting process is found to converge weakly to the Wiener Brownian motion, independently of H and of b . Thus, to the usual additive paths toward Wiener measure, this procedure adds an infinity of multiplicative paths.

1. INTRODUCTION

To motivate and clarify a new construction, this introduction compares it with others that are widely familiar. After a non-random construction has been randomized, its outcome may range from "loosening up" slightly to changing completely. Both possibilities, as well as intermediate ones, enter in this paper. The point of departure is a family of non-random "cartoon" functions [19] that are constructed by multiplicative interpolation. Designed as counterparts of Wiener Brownian motion [24, 25] or fractional Brownian motion [14, 16], they have proven to be very useful in teaching and in applications. They, in turn, are made random in this paper, in a way that seems a "natural inverse" but actually fails to be a straightforward step back to the original. The fact that it reveals new interesting phenomena suggests that the study of fractals/multifractals continues to be in large part driven by novel special constructions with odd properties, and not only by a general theory. Those non-random cartoons, together with a few other examples, contradict the widely held belief that multifractal functions (variable Hölder's H) are constructed by "multiplicative chaos" and unifractal functions (uniform Hölder's H), by "additive chaos".

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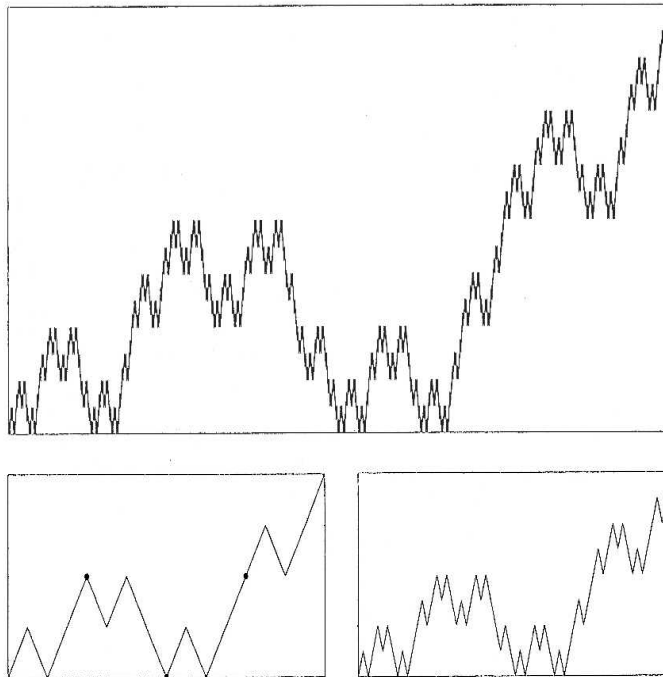


FIGURE 1. A Basic non-random cartoon of Wiener Brownian motion. Stage 2 (lower left). Stage 3 (lower right). Stage 4 (top panel).

The non-random prototype described in [19] is the crudest cartoon of Wiener-Brownian motion illustrated in figure 1. The indicator joins the points $(0,0)$ and $(1,1)$. The base is $b = 4$ and the generator $G(t)$ is graphed by four intervals of slope 2 or -2 forming a piecewise linear continuous graph linking the following points: $(0,0)$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(\frac{3}{4}, \frac{1}{2})$, and $(1,1)$. Recursive interpolation using this generator yields a curve characterized by the Fickian exponent $H = \frac{1}{2}$.

A very limited randomization, described as "shuffling", moves the interval of slope -2 along the abscissa from the second position to a randomly chosen position. Shuffling is a familiar step in binomial or multinomial multifractal measures. More interesting is the more thorough randomization introduced in [17] and called "canonical". In this context it chooses each of the four intervals of the generator at random, independently of the others, so that increasing and decreasing intervals have probabilities equal to their frequencies in the original cartoon. Here $p_+ = \frac{3}{4}$ and $p_- = \frac{1}{4}$. The increment $G(1) - G(0)$ is no longer equal to 1, but random with the expected value 1. As a result, the construction is no longer a recursive interpolation and can be called a recursive refinement.

A more general construction of a non-random cartoon has an arbitrary base $b > 3$ and a continuous piecewise linear generator made of b intervals of slope $\frac{b}{c}$, where $c \leq b$ is a second integer base. In that case, defining H by $c = b^H$ and p as $p = b^{H-1}$, the frequencies f_+ and f_- are $f_+ = \frac{1}{2} + \frac{1}{2} \frac{c}{b} = \frac{1}{2} + \frac{b^{H-1}}{2} = \frac{1}{2} + \frac{p}{2}$ and

$f_- = \frac{1}{2} - \frac{p}{2}$. The limit "cartoon" is of unbounded variation and shares the Hölder and Hausdorff properties of Wiener or Fractional Brownian Motion with the same H . In the Brownian motions and their cartoons, the correlations between past and future is known to take the form $2^{2H-1} - 1$. It is positive $\in (0, 1)$ in the persistent case $H > \frac{1}{2}$, and negative $\in (-\frac{1}{2}, 0)$ in the antipersistent case $H < \frac{1}{2}$.

A canonical randomization of any of the Brownian cartoons can now be described. A first step consists in making all those frequencies into probabilities. A second step consists in eliminating various constraints on H and p that are due to their origin in cartoons. Both the Wiener and Fractional Brownian motions non-random cartoons require that $0 < H < 1$ and that H be a ratio of logarithms of integers, of the form $\frac{\log c}{\log b}$. We shall allow p to vary from 0 to 1, which implies $-\infty < H < 1$, and leave c unrestricted, allowing it even to be smaller than 1.

This paper's object is to describe the limits of the functions $\{B_n^H\}_{n \geq 1}$ generated by this procedure. The derivatives of the functions $\{B_n^H\}_{n \geq 1}$ (in the sense of distributions) form a signed measure-valued martingale. Moreover, B_n^H is absolutely continuous, and the correlation of the derivatives of B_n^H and B_{n+1}^H is equal to p almost everywhere. For the classic positive canonical cascades [17], those martingales converge strongly to a limit. But that limit can degenerate to 0, and, if so, no normalization yielding a non zero limit is known.

For the bilateral canonical cascades considered in this paper, the situation will be shown to be altogether different, following a pattern first observed in exploratory simulations. The persistent case $\frac{1}{2} < H < 1$ behaves as expected, as illustrated in Figures 2 and 3: the martingale converges and has a strong limit, namely, a non-Gaussian variant process sharing the Hölder and Hausdorff properties of the persistent Fractional Brownian Motion of exponent H (Theorem 5.1). The antipersistent case $-\infty < H \leq \frac{1}{2}$, to the contrary, defies facile extrapolation. As illustrated in Figures 4, 5 and 6, the martingale does not converge to zero but oscillates increasingly wildly. However, there does exist an alternative normalization that yields a nondegenerate weak limit for $n \rightarrow \infty$, namely, Wiener Brownian Motion. The fact that the exponent H no longer affects the limit is a surprising form of what physicists call "universality", a phenomenon that recalls the Gaussian central limit theorem. It expresses that the rules of the cascade are destroyed in the limit, leaving only an accumulation of noise. Our result provides new functional central limit theorems. Moreover, the normalization factor in the special case $H = 1/2$ is atypical (see Theorem 3.2 and Corollary 4.2). Since H is no longer a Hölder, negative values of H create no paradox whatsoever.

To ensure a long-range power-law correlation function, exquisite long range order must be present in Fractional Brownian cartoons. Under canonical randomization this order is robust in the case of persistence. But it is non robust and destroyed in the case of antipersistence, with a clear critical point in the Wiener Brownian case.

This is novel but recalls an observation concerning the Cauchy-Lévy stable exponent α : it is constrained to $0 \leq \alpha < 2$, that is, $H > \frac{1}{2}$. The non-random cartoon of one such process [18] has a generator joining the points $(0, 0)$, $(\frac{1}{2}, p)$, $(\frac{1}{2}, 1 - p)$, $(1, 1)$. For all $p \in (0, 1)$, this can be interpolated into a discontinuous function in which the discontinuities Δ have a distribution of the form $\Pr\{\Delta > \delta\} \sim \delta^{-\alpha}$ with $\alpha = \frac{-1}{\log_2 p}$. This exponent can range as $\alpha \in (0, \infty)$. But let us, after k stages, change the number of discontinuities of given size and the number of continuous

steps. Rather than fixed, let them be random Poissonian with the same expectation. When $\alpha < 2$, the process converges to a stable one, but when $\alpha > 2$, it explodes.

In any event, for $H < \frac{1}{2}$ the integral of the covariance of the Fractional Brownian Motion vanishes. This demands very special correlation properties that are easily destroyed by diverse manipulations. "Instability" also characterizes for $H < \frac{1}{2}$ the limits of expressions of the form $\Sigma G[\Delta B_H(t)]$, where $\Delta B_H(t)$ is the increment of a Fractional Brownian Motion over $[t, t+1]$ and G is a strongly non linear transform [23]. In this context, the fact that Kolmogorov's turbulence takes on the unstable value $H = \frac{1}{3}$ may reward a close look.

In the preceding first-approximation results, we perceived an analogy with the usual sums of iid random variables X of finite variance. The strong law of large numbers tells us that the sample average, a normalized sum of n variables X_k , strongly converges to $\mathbb{E}(X)$. Then the central limit theorem tells us that the discrepancy between $\mathbb{E}(X)$ and the n th normalized sum can be subjected to a different normalization — division by \sqrt{n} — and after that converges weakly to a Wiener Brownian motion. When $H \neq 1/2$, this situation generalizes to our canonical cascades, but with some major changes. Here, $\mathbb{E}(X)$ is replaced for $H > \frac{1}{2}$ by a variant fractional Brownian motion and for $H < \frac{1}{2}$ by 0. The rate of convergence for the remainder depends on H . It takes the same analytic form $b^{n(\frac{1}{2}-H)}$ for all $H \neq 1/2$ but plays different roles: For $H < \frac{1}{2}$ it compensates for boundless growth, and for $H > 1/2$, for a decrease to 0 (Theorems 3.1 and 6.1).

Our construction possesses a natural extension to the case $H = -\infty$ if we consider the processes B_n^H/b^{-nH} and let H tend to $-\infty$ for every $n \geq 1$. In this case, $p = 0$ and k recursions yield b^k values of a random walk with symmetric correlated increments taking values in $\{-1, 1\}$. After division by $b^{\frac{k}{2}}$, there is a limit in distribution for the associated piecewise linear function, namely, the Wiener Brownian Motion (see Corollary 4.1). This is an unexpected extension of the usual result known for the classical random walk obtained by coin tossing. Here, convergence is as "weak" as can be, since the terms in the sequence are statistically independent.

Understanding bilateral cascades is helped by a step that has been fruitful since the earliest canonical cascades [17]. It consists in keeping p constant, replacing the interval $[0, 1]$ by the cube $[0, 1]^E$, and varying the Euclidean dimension E from a large value down. In all cascade constructions, the proper distance is not Euclidean but ultrametric. Hence, the nondegenerate versus degenerate alternative requires no new argument: it proceeds just as on a linear grid of base b^E . The critical $H = \frac{1}{2}$ now corresponds to $p_{crit} = b^{\frac{-E}{2}}$, which $\searrow 0$ as $E \nearrow \infty$. In a high-dimensional space, a cascade with the given p yields a variant fractional Wiener signed measure but the intersections of that measure by subspaces of small E degenerates to an infinitesimal Wiener measure (this extension to higher dimensions will be studied in a further work). Classically, this is also the case in birth and death cascades with multiplier values 1 and 0. The novelty present in the bilateral case is that the term "degenerate" takes a different meaning.

The martingales considered in this paper are the very simplest special case of the following more general construction. As for positive canonical cascades, given an integer $b \geq 2$, the recursive process consists in associating with each b -adic subinterval J of $[0, 1]$ a random weight W_J so that these weights are i.i.d with a random variable W and $\mathbb{E}(W)$ is defined and equal to $1/b$. Then, one gets a sequence

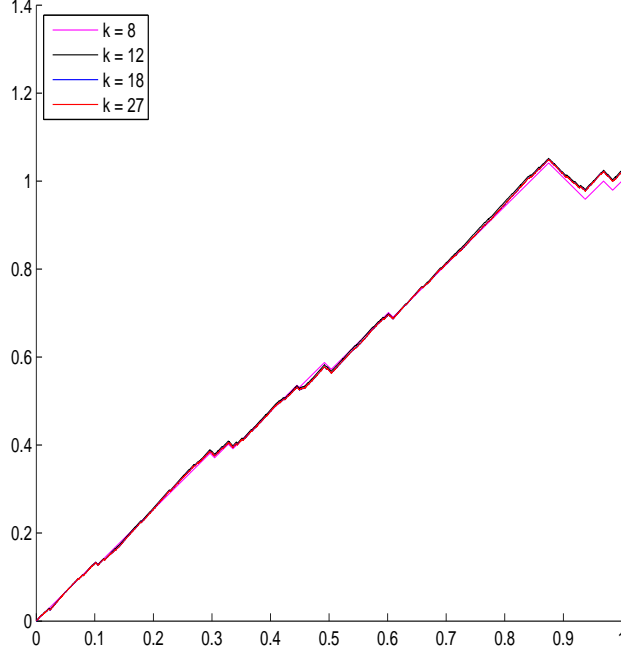


FIGURE 2. B_k^H for $k = 8, 12, 18, 27$ in the case $b = 2$ and $H = 0.95$: Fast strong convergence.

of random piecewise functions $(F_n)_{n \geq 1}$ by imposing that $F_n(0) = 0$ and that the increment of F_n over the interval J of the n^{th} generation is equal to the product $W_{J_1} W_{J_2} \cdots W_{J_n}$, where J_k is the b -adic interval of generation k containing $J (= J_n)$. Observe that this construction falls in the category of infinite products of functions [10, 2]. The family $\{F_n\}_{n \geq 1}$ forms a continuous functions-valued martingale. A sufficient condition for the sequence F_n to converge almost surely uniformly is that the function $\tau_W(p) = q - 1 - \log_b \mathbb{E}(|W|^p)$ takes a positive value at some $p \in (1, 2]$. In the simplest case studied in this paper, W belongs to $\{-b^{-H}, b^{-H}\}$, and the critical value $H = 1/2$ separates the domain $H \leq 1/2$ for which $\tau_W(p) \leq 0$ over $[1, 2]$ and the domain $H \in (1/2, 1]$ for which we always have $\tau_W(2) > 0$. In the general case, when $\tau_W((1, 2]) \not\subset (-\infty, 0]$, the limit of the signed canonical cascade is not a unifractal but a multifractal function – to be studied in a further work.

The rest of the paper is organized as follows. This section ends with definitions and notations used in the sequel. Also, the processes studied in this paper are more formally defined than in the previous paragraphs. Section 3 and 5 provide our main results for the cases $p \leq b^{-1/2}$ (i.e. $H \leq 1/2$) and $p > b^{-1/2}$ (i.e. $H > 1/2$) respectively. The next three sections are devoted to the proofs of our main results.

2. CONSTRUCTION OF THE MARTINGALE

2.1. Definitions and notations.

Let $b \geq 2$ be an integer.

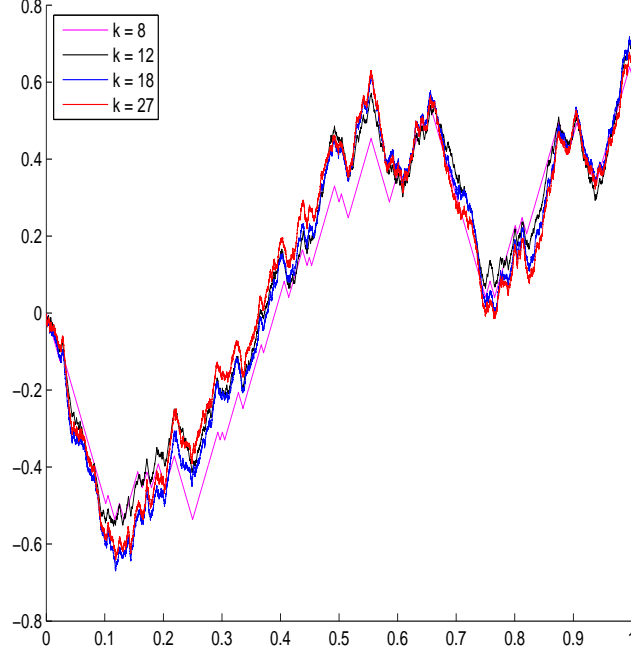


FIGURE 3. B_k^H for $k = 8, 12, 18, 27$ in the case $b = 2$ and $H = 0.7$. Strong convergence.

For $n \geq 0$ let $\Sigma_n = \{0, \dots, b-1\}^n$, where Σ_0 contains only the empty word denoted by \emptyset . Also let $\Sigma^* = \bigcup_{n \geq 0} \Sigma_n$ and $\Sigma = \{0, \dots, b-1\}^{\mathbb{N}^*}$. The concatenation operation from $\Sigma^* \times (\Sigma^* \cup \Sigma)$ to $(\Sigma^* \cup \Sigma)$ is denoted \cdot .

For $x \in \Sigma$ and $n \geq 1$, let $x|n$ be the projection of x on Σ_n and $x|\infty = x$. Then for $n \geq 1$ and $w \in \Sigma_n$, we set $[w] = \{x \in \Sigma : x|n = w\}$. Given two words of infinite length $x, y \in \Sigma$, one defines $x \wedge y$ as $x|n_0$, where $n_0 = \sup\{n \geq 1 : x|n = y|n\}$. Adopt the convention that $\inf \emptyset = 0$ and $x|0$ is the empty word \emptyset .

The length of any element w of Σ_n is equal to n and is denoted by $|w|$.

Denote by π the mapping $x \in \Sigma \mapsto \sum_{k=1}^{\infty} x_k b^{-k}$.

If $w \in \Sigma^*$, t_w stands for the number $\sum_{k=1}^{|w|} w_k b^{-k}$ and I_w stands for the closed b -adic interval $\pi([w])$.

For $n \geq 0$ denote by \mathcal{D}_n the set of b -adic numbers of the n^{th} generation in $[0, 1]$. Also denote by \mathcal{D} the set of all b -adic numbers of $[0, 1]$.

Denote by $\mathcal{C}([0, 1])$ the space of real valued continuous functions on $[0, 1]$. Then, for $\alpha \in (0, 1]$, $\mathcal{C}^\alpha([0, 1])$ stands for the subspace of $\mathcal{C}([0, 1])$ whose elements are uniformly α -Hölder continuous, i.e. $f \in \mathcal{C}^\alpha([0, 1])$ if and only if there exists $C > 0$ such that $|f(t) - f(s)| \leq C|t - s|^\alpha$ for all $t, s \in [0, 1]$.

If $f \in \mathcal{C}([0, 1])$, denote its modulus of continuity by $\omega(f, \cdot)$ (for $\delta \in [0, 1]$, $\omega(f, \delta) = \sup_{t, s \in [0, 1], |t-s| \leq \delta} |f(t) - f(s)|$).

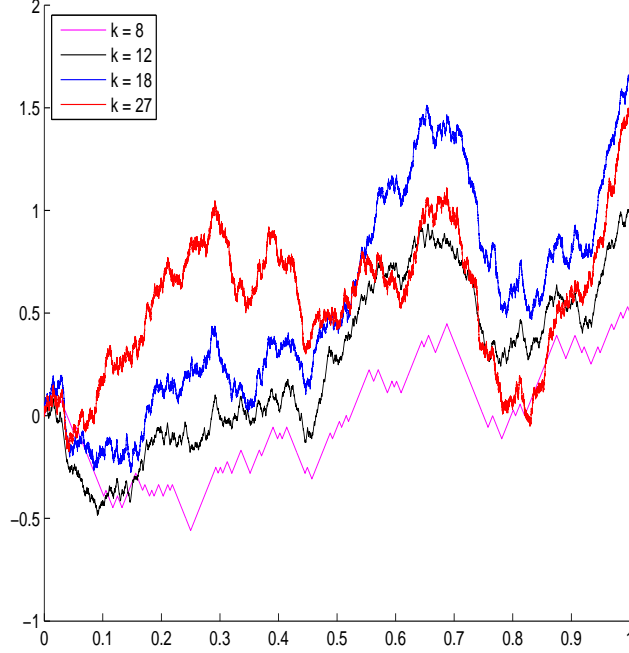


FIGURE 4. $B_k^H / \sigma_{1/2} \sqrt{k}$ for $k = 8, 12, 18, 27$ in the case $b = 2$ and $H = 0.5$: Convergence in distribution to the Wiener Brownian motion.

Recall that the pointwise Hölder exponent of f at $t_0 \in [0, 1]$ is defined by

$$h_f(t_0) = \sup \left\{ \alpha \geq 0 : \exists P \in \mathbb{R}[X], \sup_{t \in [0, 1] \setminus \{t_0\}} \frac{|f(t) - f(t_0) - P(t)|}{|t - t_0|^\alpha} < \infty \right\}.$$

If I is a subinterval of $[0, 1]$, $\Delta f(I)$ stands for $|f(\sup(I)) - f(\inf(I))|$.

2.2. A construction of a recursive canonical cascade with values ± 1 .

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space on which the random variables in the sequel are defined. If Y is a random variable, we shall denote by $\mathcal{L}(Y)$ its probability distribution.

For $0 \leq k \leq b-1$ let $S_k(t) = b^{-1}t + k$.

If $H \in [-\infty, 1]$, define the probability measure $\pi_{b,H} = p_{b,H}^+ \delta_1 + p_{b,H}^- \delta_{-1}$, where

$$p_{b,H}^+ = \frac{1 + b^{H-1}}{2} \quad \text{and} \quad p_{b,H}^- = 1 - p_{b,H}^+,$$

with the convention $b^{-\infty} = 0$.

For all $H \in [-\infty, 1]$, let $(\epsilon^H(w))_{w \in \Sigma^*}$ be a sequence of mutually independent random variables of common probability distribution $\pi_{b,H}$. When H is fixed in the sequel, sometimes we simply write $\epsilon(w)$ for $\epsilon^H(w)$.

If $u \in [0, 1] \setminus \mathcal{D}$, we identify u with the unique element $\tilde{u} \in \Sigma$ such that $u = \pi(\tilde{u})$. Then, for every $n \geq 1$, if $H \in (-\infty, 1]$ we consider on $[0, 1]$ the continuous piecewise linear map B_n^H over the b -adic intervals of the n^{th} generation such that $B_n^H(0) = 0$

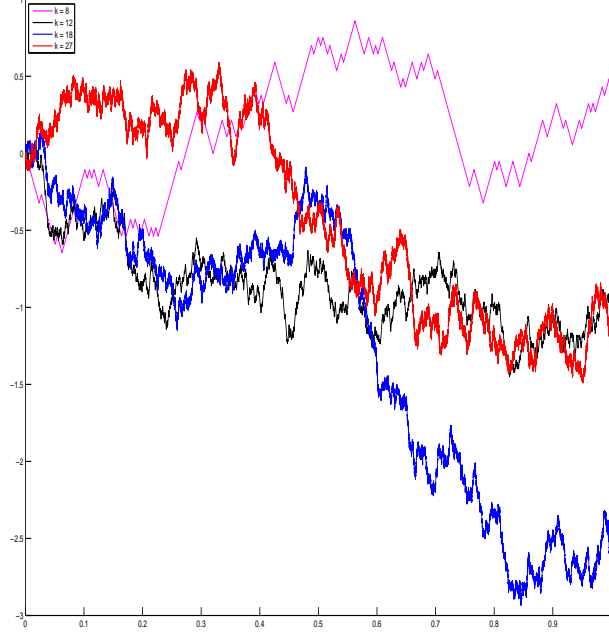


FIGURE 5. $B_k^H / \sigma_H b^{k(1/2-H)}$ for $k = 8, 12, 18, 27$ in the case $b = 2$ and $H = 0.25$: Convergence in distribution to the Wiener Brownian motion.

and for $w \in \Sigma_n$ the increment of B_n^H over I_w is equal to $\epsilon^H(w|1) \cdots \epsilon^H(w|n) b^{-nH}$, i.e.

$$B_n^H(t) = b^{-nH} \int_0^t b^n \epsilon^H(u|1) \cdots \epsilon^H(u|n) du,$$

We leave the reader verify that the sequence $(B_n^H)_{n \geq 1}$ is a $\mathcal{C}([0, 1])$ -valued martingale with respect to the filtration $(\sigma(\epsilon(w) : w \in \Sigma_n))_{n \geq 1}$.

More generally for all $w \in \Sigma^*$ let

$$B_n^H(w)(t) = b^{-nH} \int_0^t b^n \epsilon^H(w \cdot u|1) \cdots \epsilon^H(w \cdot u|n) du.$$

Of course, $B_n^H(\emptyset) = B_n^H$ almost surely.

For all $0 \leq k \leq b-1$ and $t \in I_k = [k/b, (k+1)/b]$ we have the relation

$$(2.1) \quad B_n^H(t) - B_n^H(k/b) = \epsilon^H(k) b^{-H} B_{n-1}^H(k) (S_k^{-1}(t))$$

and more generally for all $w \in \Sigma^*$, $t \in I_w$ and $n \geq |w|$

$$(2.2) \quad B_n^H(t) - B_n^H(t_w) = \epsilon^H(w|1) \cdots \epsilon^H(w|n) b^{-nH} B_{n-|w|}^H(w) (S_{w_n}^{-1} \circ \cdots \circ S_{w_1}^{-1}(t)).$$

For $n \geq 0$ and $w \in \Sigma^*$ we denote by $Z_n(w)$ the random variable $B_n^H(w)(1)$, with the convention $B_0^H(w)(1) = 1$. When $w = \emptyset$, we simply write Z_n for $Z_n(w)$. The

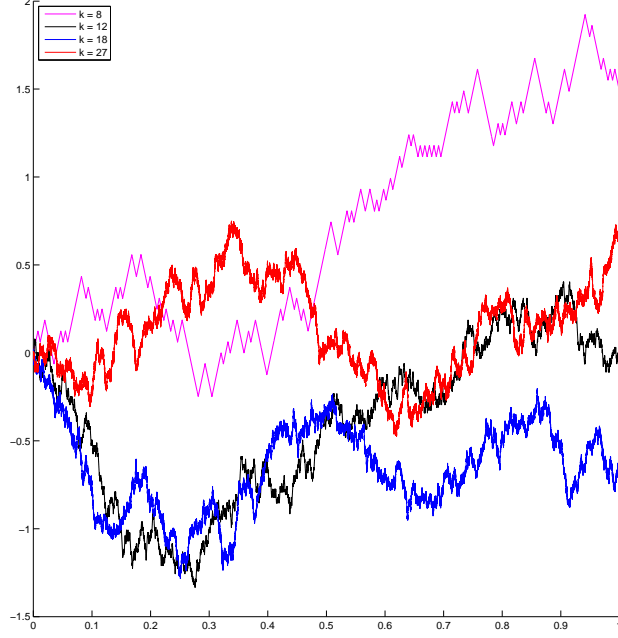


FIGURE 6. $B_k^H / \sigma_H b^{k(1/2-H)}$ for $k = 8, 12, 18, 27$ in the case $b = 2$ and $H = -2$: Convergence in distribution to the Wiener Brownian motion.

relation (2.1) yields for every $n \geq 1$

$$(2.3) \quad Z_n = \sum_{k=0}^{b-1} b^{-H} \epsilon_k Z_{n-1}(k),$$

where the random variables $\epsilon_0, \dots, \epsilon_{b-1}, Z_{n-1}(0), \dots, Z_{n-1}(b-1)$ are mutually independent. Moreover, $\mathcal{L}(\epsilon_k) = \pi_{b,H}$ and $\mathcal{L}(Z_{n-1}(k)) = \mathcal{L}(Z_{n-1})$ for all $0 \leq k \leq b-1$. Relation (2.3), which will be useful in the sequel, is familiar from the positive cascade case [17].

Finally, for $H \in [-\infty, 1/2]$ let

$$\sigma_H = \begin{cases} 1 & \text{if } H = -\infty \\ \sqrt{\frac{b-1}{b^{2-2H}-b}} + 1 & \text{if } H \in (-\infty, 1/2) \\ \sqrt{\frac{b-1}{b}} & \text{if } H = 1/2 \end{cases}.$$

3. WEAK CONVERGENCE OF THE NORMALIZED MARTINGALE B_n^H TO WIENER BROWNIAN MOTION, INDEPENDENTLY OF b AND H IN THE ANTI-PERSISTANT CASE $H \leq 1/2$

Theorem 3.1. *Let $H \in (-\infty, 1/2)$. The sequence $(\mathcal{L}(B_n^H/\sigma_H b^{n(1/2-H)}))_{n \geq 1}$ converges weakly to the Wiener measure as n goes to ∞ .*

Theorem 3.2. *The sequence $(\mathcal{L}(B_n^{1/2}/\sigma_{1/2}\sqrt{n}))_{n \geq 1}$ converges weakly to the Wiener measure as n goes to ∞ .*

Remark 3.1. (1) When $H < 1/2$, Theorem 3.1 implies that with probability 1, $\limsup_{n \rightarrow \infty} \frac{\|B_n^H\|_\infty}{b^{n(1/2-H)}} > 0$. Thus the martingale B_n^H neither strongly converges to a non trivial limit in $\mathcal{C}([0, 1])$ nor to 0. This fact deserves to be called degeneracy. This shows a strong difference with positive canonical cascades for which degeneracy means uniform convergence to 0 ([17, 12]). The same remarks hold when $H = 1/2$.

(2) For $H \leq 1/2$ define $X_n^H = B_n^H(1)/\sigma_H b^{n(1/2-H)}$ if $H < 1/2$ and $X_n^H = B_n^{1/2}(1)/\sigma_{1/2}\sqrt{n}$ otherwise. The reader can check that $X_n^H(1)$ is not a Cauchy sequence in L^2 while the L^2 norm of $X_n^H(1)$ converges to 1 as n goes to ∞ . This implies that $X_n^H(1)$ cannot converge almost surely to a standard normal random variable. Consequently, Theorem 3.1 and 3.2 cannot be strengthened into results of almost sure convergence.

4. RESTATEMENT OF THEOREMS 3.1 AND 3.2 AS FUNCTIONAL CLT WITH ATYPICAL NORMALIZATION WHEN $H = 1/2$

If $H \in [-\infty, 1/2]$, $n \geq 1$ and $0 \leq k < b^n$ and w is the unique element of Σ_n such that $t_w = kb^{-n}$ let

$$\xi_k^{(n,H)} = \prod_{j=1}^n \epsilon^H(w|j).$$

For a given $n \geq 1$, the random variables $\xi_k^{(n,H)}$, $0 \leq k < b^n$, are identically distributed, and they take values in $\{-1, 1\}$.

Also, consider the random walk $(S_r^{(n,H)})_{0 \leq r < b^n}$ defined by

$$S_r^{(n,H)} = \sum_{k=0}^{r-1} \xi_k^{(n,H)}$$

(with the convention $S_{-1}^{(n,H)} = 0$).

Corollary 4.1 (Functional central limit theorem). *Let $H \in [-\infty, 1/2]$ and for $n \geq 1$ and $t \in [0, 1]$ define $X_n^H(t) = \frac{1}{\sigma_H \sqrt{b^n}} [S_{[b^n t]}^{(n,H)} + (b^n t - [b^n t])\xi_{[b^n t]}^{(n,H)}]$. The sequence $\mathcal{L}(X_n^H)_{n \geq 1}$ converges weakly to the Wiener measure as n tends to ∞ .*

Corollary 4.2 (Functional central limit theorem). *For $n \geq 1$ and $t \in [0, 1]$ define $X_n^{1/2}(t) = \frac{1}{\sigma_{1/2} \sqrt{nb^n}} [S_{[b^n t]}^{(n,1/2)} + (b^n t - [b^n t])\xi_{[b^n t]}^{(n,1/2)}]$. The sequence $\mathcal{L}(X_n^{1/2})_{n \geq 1}$ converges weakly to the Wiener measure as n tends to ∞ .*

Remark 4.1. (1) When $H < 1/2$, the stochastic process X_n^H takes formally the same form as the processes considered in central limit theorems for weakly dependent sequences (see [4], Ch. 19 or [7]). The main difference is that in the process we consider the random variables $\xi_k^{(n,H)}$ are highly correlated. Nevertheless, the same asymptotic behavior (weak convergence to the Wiener measure) holds.

(2) In the case $H = 1/2$, the normalizing factor takes an untypical form.

Remark 4.2. When $H = -\infty$, it is not difficult to verify that the conclusions of Propositions 7.1–7.3 hold for $(X_n^{-\infty})_{n \geq 1}$ instead of the sequence $(B_n^H/a_n^H)_{n \geq 1}$ considered in Section 7 by using the same approach. Consequently, the proof is left to the reader in this case.

Remark 4.3. We mention that functional central limit theorems associated with positive canonical cascades have been established in [15] in a very different spirit. There, a square integrable random weight W is fixed which generates a canonical multiplicative cascades in base b and its associated sequence of increasing functions $F_n^{(b)}$ on $[0, 1]$ converging to a function $F_\infty^{(b)}$. For each $n \in \mathbb{N}^* \cup \{\infty\}$ the authors establish a functional central limit theorem for $(F_n^{(b)}(t) - t)/\sqrt{b}$ as the basis b tends to ∞ : $(F_n^{(b)}(t) - t)/\sqrt{b}$ converges in law to a multiple of the Brownian motion. As letting b tend to ∞ weakens the correlations between the increments of $F^{(b)}$, the existence of such a weak limit is natural.

5. STRONG CONVERGENCE OF THE MARTINGALE IN THE PERSISTENT CASE

$$1/2 < H \leq 1$$

Theorem 5.1. Suppose that $H \in (1/2, 1]$. The sequence $(B_n^H)_{n \geq 1}$ is a martingale that converges almost surely and in L^2 norm to a continuous function B^H . Moreover, with probability 1,

- (1) B^H belongs to $\bigcap_{H' < H} C^{H'}([0, 1])$ and it has everywhere a pointwise Hölder exponent equal to H .
- (2) The Hausdorff and box dimensions of the graph of B^H is $2 - H$.

Remark 5.1. The limit process B^H is not Gaussian since a computation shows that the third moment of the centered random variable $B^H(1) - 1$ does not vanish. Notice that the case $H = 1$ yields the deterministic function $B^1(t) = t$.

6. FUNCTIONAL CLT ASSOCIATED WITH THE STRONG CONVERGENCE CASE

$$1/2 < H < 1$$

It will be shown in Section 8 that $\mathbb{E}(B^H(1)^2) < \infty$ if $H > 1/2$. Consequently the number $\sigma_H = \sqrt{\mathbb{E}(B^H(1)^2) - 1}$ is positive and finite when $H \in (1/2, 1)$.

The following Theorems 6.1 and 6.2 must be viewed as counterparts of Theorems 3.1 and Corollary 4.1.

Theorem 6.1. Let $H \in (1/2, 1)$. The sequence $((B^H - B_n^H)/\sigma_H b^{n(1/2-H)})_{n \geq 1}$ converges weakly to the Wiener measure as n tends to ∞ .

If $H \in (1/2, 1)$, for every $w \in \Sigma^*$ denote by $B^H(w)$ the almost sure limit of $B_n^H(w)$. Also, if $n \geq 1$ and $0 \leq k < b^n$ and w is the unique element of Σ_n such that $t_w = kb^{-n}$ let

$$\tilde{\xi}_k^{(n,H)} = (B^H(w)(1) - 1) \prod_{j=1}^n \epsilon^H(w|j).$$

Then define $S_p^{(n,H)} = \sum_{k=0}^{p-1} \tilde{\xi}_k^{(n,H)}$ for $0 \leq p < b^n$ and finally consider on $[0, 1]$ the piecewise linear function

$$X_n^H(t) = \frac{1}{\sigma_H \sqrt{b^n}} \left[S_{[b^n t]}^{(n,H)} + (b^n t - [b^n t]) \tilde{\xi}_{[b^n t]}^{(n,H)} \right].$$

Theorem 6.2. *Let $H \in (1/2, 1)$. The sequence $\mathcal{L}(X_n^H)_{n \geq 1}$ converges weakly to the Wiener measure as n tends to ∞ .*

Remark 6.1. *Theorem 6.1 implies that $(B^H - B_n^H)(1)/\sigma_H b^{n(1/2-H)}$ converges in law to a $\mathcal{N}(0, 1)$ law. This result is of the same nature as Proposition 4.1 in [22] which deals with central limits theorems associated with non negative canonical cascades. The technique used in [22] would work to establish the convergence of $\mathcal{L}((B^H - B_n^H)(1)/\sigma_H b^{n(1/2-H)})$. It uses Lindeberg's theorem, while we exploit the functional equation (2.3).*

7. PROOF OF THEOREM 3.1 AND 3.2 AND THEIR COROLLARIES CONCERNING THE CASE $H \leq \frac{1}{2}$, I.E., $p \leq b^{-1/2}$

Theorems 3.1 and 3.2 follow from the next three propositions. In fact we are going to show that the sequence $(\mathcal{L}(B_n^H/a_n^H))_{n \geq 1}$ converges weakly to the Wiener

measure, where for $n \geq 1$ $a_n^H = \sqrt{\left(\frac{b-1}{b} + b^{1-2H} - 1\right) \frac{b^{n(1-2H)} - 1}{b^{1-2H} - 1}}$ if $H < 1/2$

and $a_n^H = \sqrt{\frac{b-1}{b}} n$ if $H = 1/2$ (observe that by the L'Hospital rule, a_n^H converges to $a_n^{1/2}$ as $H \nearrow 1/2$). It is easily seen that this will imply Theorems 3.1 and 3.2 and so their corollaries Corollaries 4.1 and 4.2.

When $H < 1/2$, the normalization by a_n^H is more practical to use than $\sigma_H b^{n(1/2-H)}$ because it naturally appears in the B_n^H asymptotic behavior's study.

Proposition 7.1. *Let $H \in (-\infty, 1/2]$. The sequence $(\mathcal{L}(B_n^H(1)/a_n^H))_{n \geq 1}$ converges to $\mathcal{N}(0, 1)$ as n goes to ∞ .*

Proof. Let $Y_n = B_n^H(1)/a_n^H = Z_n/a_n^H$. It is enough to show that

- (1) for every $p \geq 0$ one has the property (\mathcal{P}_{2p}) : $M_{2p} = \lim_{n \rightarrow \infty} \mathbb{E}(Y_n^{2p})$ exists. Moreover $M_2 = 1$;
- (2) for every $p \geq 0$ one has the property (\mathcal{P}_{2p+1}) : $\lim_{n \rightarrow \infty} \mathbb{E}(Y_n^{2p+1}) = 0$;
- (3) the moments of even orders obey the following induction relation valid for $p \geq 2$:

$$M_{2p} = (b^p - b)^{-1} \sum_{\substack{0 \leq \alpha_0, \dots, \alpha_{b-1} < p \\ \sum_{k=0}^{b-1} \alpha_k = p}} \frac{(2p)!}{(2\alpha_0)! \cdots (2\alpha_{b-1})!} \prod_{k=0}^{b-1} M_{2\alpha_k}.$$

Indeed, (1) will ensure that the sequence of probability distributions $\mathcal{L}(Y_n)$ is tight. Moreover, it is easy to verify that a $\mathcal{N}(0, 1)$ random variable N is so that its moments of even orders satisfy the same relation as the numbers M_{2p} , $p \geq 1$, defined by $M_2 = 1$ and the induction relation (3) (to see this, write N as the sum of

b independent $\mathcal{N}(0, b^{-1/2})$ random variables). Consequently, since the law $\mathcal{N}(0, 1)$ is characterized by its moments, Y_n must converge in law to $\mathcal{N}(0, 1)$.

Let us establish (1), (2) and (3).

Let us take the expectation of the square of (2.3), by using the fact that $\mathbb{E}(Z_n) = 1$. This will explain the introduction of the normalization factor a_n^H .
We have (notice that $\mathbb{E}(\epsilon_0) = b^{H-1}$)

$$(7.1) \quad \mathbb{E}(Z_n^2) = b^{1-2H} \mathbb{E}(Z_{n-1}^2) + b(b-1)b^{-2H} (\mathbb{E}(\epsilon_0))^2$$

$$(7.2) \quad = b^{1-2H} \mathbb{E}(Z_{n-1}^2) + \frac{b-1}{b}.$$

This yields $\mathbb{E}(Z_n^2) = (a_n^H)^2 + 1$. In particular, the limit M_2 is well defined and equals 1. Moreover, $\lim_{n \rightarrow \infty} \mathbb{E}(Y_n) = 0$ since $\mathbb{E}(Z_n) = 1$ and $\lim_{n \rightarrow \infty} a_n^H = \infty$.

Now let q be an integer ≥ 3 . Taking the expectation of (2.1) to the power q yields

$$(7.3) \quad \mathbb{E}(Z_{n+1}^q) = b^{1-qH} \mathbb{E}(\epsilon_0^q) \mathbb{E}(Z_n^q) + b^{-qH} \sum_{\substack{0 \leq \beta_0, \dots, \beta_{b-1} < q \\ \sum_{k=0}^{b-1} \beta_k = q}} \gamma_{(\beta_0, \dots, \beta_{b-1})} \prod_{k=0}^{b-1} \mathbb{E}(\epsilon_0^{\beta_k}) \mathbb{E}(Z_n^{\beta_k}),$$

where $\gamma_{\beta_0, \dots, \beta_{b-1}} = \frac{q!}{(\beta_0)! \dots (\beta_{b-1})!}$.

Let us denote $\mathbb{E}(Y_n^q)$ by $M_q^{(n)}$, the set $\{(\beta_0, \dots, \beta_{b-1}) \in \mathbb{N}^b : 0 \leq \beta_0, \dots, \beta_{b-1} < q, \sum_{k=0}^{b-1} \beta_k = q\}$ by S_q , the ratio $\sqrt{\frac{n}{n+1}}$ by $r_n^{(1/2)}$ and the ratio $\sqrt{\frac{b^{n(1-2H)} - 1}{b^{(n+1)(1-2H)} - 1}}$ by $r_n^{(H)}$ when $H < 1/2$.

Now, using that $\mathbb{E}(\epsilon_0^q) = b^{H-1}$ or 1 depending on q is an odd or an even number, (7.3) yields for $H \leq 1/2$

$$(7.4) \quad M_q^{(n+1)} = \begin{cases} (r_n^{(H)})^q \left(b^{-(q-1)H} M_q^{(n)} + b^{-qH} \sum_{\beta \in S_q} \gamma_{\beta} \prod_{k=0}^{b-1} \mathbb{E}(\epsilon_0^{\beta_k}) M_{\beta_k}^{(n)} \right) & \text{if } q \text{ is odd,} \\ (r_n^{(H)})^q \left(b^{1-qH} M_q^{(n)} + b^{-qH} \sum_{\beta \in S_q} \gamma_{\beta} \prod_{k=0}^{b-1} \mathbb{E}(\epsilon_0^{\beta_k}) M_{\beta_k}^{(n)} \right) & \text{if } q \text{ is even} \end{cases}.$$

We show by induction that $((\mathcal{P}_{2p-1}), (\mathcal{P}_{2p}))$ holds for $p \geq 1$, and we deduce the relation (3).

We have shown that $((\mathcal{P}_1), (\mathcal{P}_2))$ holds. Suppose that $((\mathcal{P}_{2k-1}), (\mathcal{P}_{2k}))$ holds for $1 \leq k \leq p-1$, with $p \geq 2$. In particular, $M_{\beta_k}^{(n)}$ goes to 0 as n goes to ∞ if β_k is an odd integer belonging to $[1, 2p-3]$.

Suppose $H = 1/2$ and simply denote $r_n^{(1/2)}$ by r_n . Every element of the set S_{2p-1} must contain an odd component. Due to our induction assumption, this implies that in the relation (7.4), the term $r_n^{2p-1} b^{-(2p-1)/2} \sum_{\beta \in S_{2p-1}} \gamma_{\beta} \prod_{k=0}^{b-1} \mathbb{E}(\epsilon_0^{\beta_k}) M_{\beta_k}^{(n)}$

in the right hand side of $M_{2p-1}^{(n+1)}$ goes to 0 at ∞ . This yields

$$M_{2p-1}^{(n+1)} = r_n^{2p-1} b^{-(p-1)} M_{2p-1}^{(n)} + o(1)$$

as $n \rightarrow \infty$. Since $r_n^{2p-1} b^{-(p-1)} \leq b^{1-p} < 1$, this yields $\lim_{n \rightarrow \infty} M_{2p-1}^{(n)} = 0$, that is to say (\mathcal{P}_{2p-1}) .

Now, the same argument as above shows that in the right hand side of $M_{2p}^{(n+1)}$, we have

$$\lim_{n \rightarrow \infty} \sum_{\beta \in S_{2p}} \gamma_\beta \prod_{k=0}^{b-1} \mathbb{E}(\epsilon_0^{\beta_k}) M_{\beta_k}^{(n)} = \sum_{\substack{\beta \in S_{2p} \\ \beta_k \text{ even}}} \gamma_\beta \prod_{k=0}^{b-1} M_{\beta_k}.$$

Denote by L the right hand side of the above relation and define $L' = (b^p - b)^{-1} L$. By using (7.4) we deduce from the previous lines that

$$(7.5) \quad M_{2p}^{(n+1)} = r_n^{2p} b^{1-p} M_{2p}^{(n)} + b^{-p} L + o(1).$$

Then by using that $r_n \rightarrow 1$ as $n \rightarrow \infty$ and the relation $L' = b^{1-p} L' + b^{-p} L$ we obtain

$$M_{2p}^{(n+1)} - L' = r_n^{2p} b^{1-p} (M_{2p}^{(n)} - L') + o(1).$$

This yields both (\mathcal{P}_{2p}) and (3) since $r_n^{2p} b^{1-p} \sim b^{1-p} < 1$ as $n \rightarrow \infty$.

Now suppose that $H < 1/2$. Almost the same arguments as when $H = 1/2$ yield the conclusion. The only change is that we have to perform one more computation to obtain a relation equivalent to (7.5). Due to the expression of $r_n^{(H)}$, we have

$$\begin{aligned} M_{2p}^{(n+1)} &= b^{-(1-2H)p} \left(b^{(1-2pH)} M_{2p}^{(n)} + b^{-2pH} L \right) + o(1) \\ &= b^{1-p} M_{2p}^{(n)} + b^{-p} L + o(1). \end{aligned}$$

□

Proposition 7.2. *Let $H \in (-\infty, 1/2]$. Let $(W(t))_{t \in [0,1]}$ be a standard Brownian motion. For every $p \geq 1$, the probability distribution $\mathcal{L}\left((B_n^H(t)/a_n^H)_{t \in \mathcal{D}_p}\right)$ converges to $\mathcal{L}((W(t))_{t \in \mathcal{D}_p})$ as $n \rightarrow \infty$.*

Proof. Let $p \geq 1$ and denote by $0 = t_0 < t_1 < \dots < t_{2p} = 1$ the elements of \mathcal{D}_p . Also, simply denote $B_n^H(t)/a_n^H$ by $\mathcal{Y}_n(t)$ and by ϕ_n the characteristic function of $\mathcal{Y}_n(1)$ ($\mathcal{Y}_n(1)$ is nothing but the random variable Y_n studied in Proposition 7.1). By using the fact that in (2.2) the functions $B_{n-|w|}^H(w)$ are mutually independent and identically distributed with $a_{n-|w|}^H \mathcal{Y}_{n-|w|}(1)$, and also independent of the products $\epsilon(w|1) \dots \epsilon(w|n) b^{-nH}$, we can get that for $(u_w)_{w \in \Sigma_p} \in \mathbb{R}^{2^p}$ and $n > p$

$$\mathbb{E} \left(e^{i \sum_{w \in \Sigma_p} u_w (\mathcal{Y}_n(t_w + b^{-p}) - \mathcal{Y}_n(t_w))} \right) = \mathbb{E} \prod_{w \in \Sigma_p} \phi_{n-p} \left(u_w b^{-pH} \frac{a_{n-p}^H}{a_n^H} \prod_{k=1}^p \epsilon(w|k) \right).$$

It follows from Proposition 7.1 that $\phi_{n-p}(t)$ goes to $e^{-t^2/2}$ as n goes to ∞ . Moreover, $b^{-pH} a_{n-p}^H / a_n^H$ tends to $b^{-p/2}$ as n goes to ∞ . Thus, applying the dominated

convergence theorem yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left(e^{i \sum_{w \in \Sigma_p} u_w (\mathcal{Y}_n(t_w + b^{-p}) - \mathcal{Y}_n(t_w))} \right) &= \mathbb{E} \left(\prod_{w \in \Sigma_p} \exp \left(2^{-1} u_w^2 b^{-p} \prod_{k=1}^p \epsilon(w|k)^2 \right) \right) \\ &= \prod_{w \in \Sigma_p} \exp(u_w^2 b^{-p}/2) \end{aligned}$$

since the $\epsilon(w|k)$ take values in $\{-1, 1\}$. This yields the conclusion. \square

Proposition 7.3. *Let $H \in (-\infty, 1/2]$. The sequence $(\mathcal{L}(B_n^H/a_n^H))_{n \geq 1}$ of probability distributions on $C([0, 1])$ is tight.*

Proof. Let us denote by \mathcal{Y}_n the process B_n^H/a_n^H as in the proof of Proposition 7.2. By Theorem 7.3 of [4], since $\mathcal{Y}_n(0) = 0$ almost surely for all $n \geq 1$, it is enough to show that for each positive ε

$$(7.6) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\omega(\mathcal{Y}_n, \delta) \geq \varepsilon) = 0$$

(the modulus of continuity $\omega(f, \cdot)$ of a continuous function f is defined in Section 2.1).

Fix $H' \in (0, 1/2)$ and K a positive integer such that $2K(1/2 - H') > 1$. It follows from the proof of Proposition 7.1 that the sequence $(\mathbb{E}(\mathcal{Y}_m(1)^{2K}))_{m \geq 1}$ is bounded by a constant C_K . Moreover, by construction there exists a constant $C_{b,H}$ such that for all $n > p \geq 1$ we have $\frac{a_{n-p}^H}{a_n^H} \leq C_{b,H} b^{p(H-1/2)}$. By using (2.2) and a Markov inequality we can get that for $n \geq 2$, $1 \leq p < n$ and $0 \leq k \leq b^p - 1$

$$\begin{aligned} &\mathbb{P} \left(|\mathcal{Y}_n((k+1)b^{-p}) - \mathcal{Y}_n(kb^{-p})| > b^{-pH'} \right) \\ &\leq b^{2KpH'} \mathbb{E} \left(|\mathcal{Y}_n((k+1)b^{-p}) - \mathcal{Y}_n(kb^{-p})|^{2K} \right) \\ &\leq b^{2K(H'-H)p} \left(\frac{a_{n-p}^H}{a_n^H} \right)^{2K} \mathbb{E}(\mathcal{Y}_{n-p}(1)^{2K}) \\ &\leq C_K C_{b,H} b^{2K(H'-1/2)p}. \end{aligned}$$

Now let $\alpha_p = C_K C_{b,H} b^{p(1+2K(H'-1/2))}$. By our choice of H' and K the series $\sum_{p \geq 1} \alpha_p$ converge. Moreover, since for $1 \leq p < n$ the b -adic increments of generation p of \mathcal{Y}_n have the same probability distribution, we have

$$\mathbb{P} \left(\exists 0 \leq k < b^p, |\mathcal{Y}_n((k+1)b^{-p}) - \mathcal{Y}_n(kb^{-p})| > b^{-pH'} \right) \leq \alpha_p.$$

On the other hand, if $p \geq n$ and $0 \leq k < b^p$, by construction since there exists a constant $c_{b,H} < 1$ such that $a_n^H \geq c_{b,H} b^{n(1/2-H)}$ we have

$$|\mathcal{Y}_n((k+1)b^{-p}) - \mathcal{Y}_n(kb^{-p})| = \frac{b^{-n(H-1)}b^{-p}}{a_n^H} \leq \frac{b^{n/2-p}}{c_{b,H}} \leq \frac{b^{-p/2}}{c_{b,H}} \leq \frac{b^{-pH'}}{c_{b,H}}.$$

Let A_p denote the rest $\sum_{j \geq p} \alpha_j$. We deduce from the previous lines that for all $p \geq 1$,

$$\sup_{n \geq 2} \mathbb{P} \left(\exists j \geq p, \exists 0 \leq k < b^{-j}, |\mathcal{Y}_n((k+1)b^{-j}) - \mathcal{Y}_n(kb^{-j})| > c_{b,H}^{-1} b^{-jH'} \right) \leq A_p.$$

The event $\left\{ \forall j \geq p, \forall 0 \leq k < b^{-j}, |\mathcal{Y}_n((k+1)b^{-j}) - \mathcal{Y}_n(kb^{-j})| \leq c_{b,H}^{-1} b^{-jH'} \right\}$ is denoted by E_p^n . One has $\mathbb{P}(E_p^n) \geq 1 - A_p$. A simple adaptation of the proof of the Kolmogorov-Centsov theorem [5] (see the proof of Proposition 8.1(3) in the next section) shows that on E_p^n , we have

$$\sup_{n \geq 2} \sup_{\substack{0 \leq s < t \leq 1 \\ t-s \leq b^{-p}}} \frac{|\mathcal{Y}_n(t) - \mathcal{Y}_n(s)|}{(t-s)^{H'}} \leq \frac{2(b-1)c_{b,H}^{-1}}{1-b^{-H'}}.$$

Consequently, for all $n \geq 2$ we have $\omega(\mathcal{Y}_n, b^{-p}) \leq \frac{2(b-1)c_{b,H}^{-1}b^{-pH'}}{1-b^{-H'}}$. This yields

$$\inf_{n \geq 2} \mathbb{P} \left(\omega(\mathcal{Y}_n, b^{-p}) \leq \frac{2(b-1)c_{b,H}^{-1}b^{-pH'}}{1-b^{-H'}} \right) \geq \inf_{n \geq 1} \mathbb{P}(E_p^n) \geq 1 - A_p.$$

Since $\lim_{p \rightarrow \infty} A_p = 0$, the previous inequality yields (7.6). \square

Proof of Theorem 3.1. We use the notations of the three previous propositions. Suppose that $(\mathcal{Y}_{n_k})_{k \geq 1}$ is subsequence of $(\mathcal{Y}_n)_{n \geq 1}$ which converges weakly to a probability distribution \mathcal{W}_∞ . Due to Proposition 7.2, a process \mathcal{Y} such that $\mathcal{L}(\mathcal{Y}) = \mathcal{W}_\infty$ has continuous path and is such that for all $p \geq 1$, $\mathcal{L}((\mathcal{Y}(t))_{t \in \mathcal{D}_p}) = \mathcal{L}((\mathcal{W}(t))_{t \in \mathcal{D}_p})$. Since $\bigcup_{p \geq 1} \mathcal{D}_p$ is dense in $[0, 1]$ and we know that the almost sure limit of a sequence of centered Gaussian variables is a centered Gaussian variable with variance equal to the limit of the variances, we conclude that $\mathcal{W}_\infty = \mathcal{W}$. Now the final conclusion comes from Proposition 7.3.

8. PROOF OF THEOREM 5.1 CONCERNING STRONG CONVERGENCE WHEN $1/2 < H < 1$

We first construct in Proposition 8.1 a stochastic process thanks to the almost sure pointwise convergence of B_n^H over the set b -adic numbers. We establish regularity properties for this process and then identify this process as the almost sure uniform limit of B_n^H (Proposition 8.2) by using a result on vector martingales. At last we prove the result concerning the Hausdorff and box dimensions of the graph of the limit B^H of B_n^H .

Proposition 8.1. *Let $H \in (1/2, 1]$. With probability one*

- (1) *for every b -adic number t in $[0, 1]$ the sequence $B_n^H(t)$ converges to a limit denoted $B_\infty^H(t)$.*
- (2) *The function B_∞^H defined on the b -adic numbers possesses a (necessarily unique) continuous extension to $[0, 1]$ also denoted B_∞^H .*
- (3) *The function B_∞^H belongs to $C^{H'}([0, 1])$ for all $H' < H$.*
- (4) *The pointwise Hölder exponent of B_∞^H at every point of $[0, 1]$ is equal to H .*

We first establish the following useful result on the martingale $(B_n^H(1))_{n \geq 1}$.

Lemma 8.1. *Let $H \in (1/2, 1]$. The martingale $(B_n^H(1))_{n \geq 1}$ is bounded in L^q norm for all $q \geq 1$.*

Proof. Denote $B_n^H(1)$ by Z_n as in the proof of Proposition 7.1. Since $(Z_n)_{n \geq 1}$ is a martingale, the sequence $(\mathbb{E}(Z_n^2))_{n \geq 1}$ is non-decreasing. Consequently, it follows from (2.3), (7.1) and the fact that $b^{1-2H} < 1$ since $H > 1/2$ that $(Z_n)_{n \geq 1}$ is bounded in L^2 norm and thus it converges almost surely to a limit Z_∞ . Then,

the relation (7.3) as well as arguments very similar to those used in the proof of Proposition 7.1 show that the sequence $\mathbb{E}(Z_n^q)$ converges for every integer $q \geq 1$ as n goes to ∞ . In particular it is bounded in L^{2q} for every integer $q \geq 1$. This implies that $\mathbb{E}(Z_\infty^{2q}) < \infty$ for every integer $q \geq 1$ by the Fatou lemma. \square

Proof. (of Proposition 8.1) (1) Since $B_n^H(0) = 0$ for all $n \geq 1$ almost surely, it is enough to establish that for every $p \geq 1$ and $0 \leq k < b^{-p}$ the sequence $(\Delta_n^H(p, k))_{n \geq 1}$ defined by $\Delta_n^H(p, k) = B_n^H((k+1)b^{-p}) - B_n^H(kb^{-p})$ converges almost surely. Indeed, since the set of b -adic numbers is countable, this will imply that with probability one, $(\Delta_n^H(p, l))_{n \geq 1}$ converges for every, $p \geq 1$ and $0 \leq l < b^p$ as n goes to ∞ , thus $B_n^H(kb^{-p}) = \sum_{l=0}^{k-1} \Delta_n^H(p, l)$ converges for $p \geq 1$ and $1 \leq k \leq b^p$ as n goes to ∞ .

Now, it is sufficient to notice that given $p \geq 1$, $0 \leq k < b^{-p}$, and $w = w_1 \cdots w_p$ so that $kb^{-p} = \sum_{j=1}^p w_j b^{-j}$, the relation (2.2) yields for $n \geq p+1$

$$(8.1) \quad \Delta_n^H(p, k) = \epsilon(w_1) \cdots \epsilon(w_1 \cdots w_p) b^{-pH} B_{n-p}^H(w)(1).$$

The convergence of $\Delta_n^H(p, k)$ then comes from Lemma 8.1 which ensures that the martingale $(B_{n-p}^H(w)(1))_{n \geq 1}$ converges to a limit $B_\infty^H(w)(1)$ since it is bounded in L^2 -norm.

Let $\Delta_\infty^H(p, k)$ and $B_\infty^H(kb^{-p})$ denote the limit of $\Delta_n^H(p, k)$ and $B_n^H(kb^{-p})$ respectively. By construction, given $p \geq 1$, $0 \leq k < b^{-p}$, and $w = w_1 \cdots w_p$ so that $kb^{-p} = \sum_{j=1}^p w_j b^{-j}$, we have

$$(8.2) \quad \Delta_\infty^H(p, k) = B_\infty^H((k+1)b^{-p}) - B_\infty^H(kb^{-p})$$

$$(8.3) \quad = \epsilon(w_1) \cdots \epsilon(w_1 \cdots w_p) b^{-pH} B_\infty^H(w)(1).$$

(2) and (3) We adapt the proof of the Kolmogorov-Centsov theorem [5, 13] which uses the dyadic basis while we work in any basis b . Let $H' \in (0, H)$. Fix an integer $K > 1/2(H - H')$. Due to (8.2) and (8.3), for $p \geq 1$ we have

$$\alpha_p := \mathbb{P}\left(\exists 0 \leq k < b^p, |\Delta_\infty^H(p, k)| \geq b^{-pH'}\right) \leq b^{p(1+2K(H'-H))} \mathbb{E}(B_\infty^H(1)^{2K}).$$

Since $\sum_{p \geq 1} \alpha_p < \infty$, the Borel-Cantelli implies that with probability 1, there exists n_0 such that

$$(8.4) \quad \sup_{0 \leq k < b^n} |\Delta_\infty^H(n, k)| < b^{-nH'}, \quad \forall n \geq n_0.$$

Now we fix $n \geq n_0$ and show that for all $m > n$,

$$(8.5) \quad |B_\infty^H(t) - B_\infty^H(s)| \leq 2(b-1) \sum_{j=n+1}^m b^{-H'j}, \quad \forall t, s \in \mathcal{D}_m, \quad 0 < t - s < b^{-n}.$$

If $m = n+1$, one has $s = kb^{-(n+1)}$ and $t = k'-(n+1)$ with $0 < k' - k < b$, so due to (8.4) we have $|B_\infty^H(t) - B_\infty^H(s)| \leq (k' - k)b^{-(n+1)H'}$, hence the conclusion.

Suppose that (8.5) holds for $n+1 \leq m \leq M-1$. Let $t, s \in \mathcal{D}_M$ such that $0 < t - s < b^{-n}$ and consider $t_1 = \max\{u \in \mathcal{D}_{M-1} : u \leq t\}$ and $s_1 = \min\{u \in \mathcal{D}_{M-1} : u \geq s\}$. One has $s \leq s_1 \leq t_1 \leq t$, $t_1 - s_1 < b^{-n}$, $s_1 - s \leq (b-1)b^{-M}$ and $t - t_1 \leq (b-1)b^{-M}$. Now, since s_1 and t_1 belong to $\mathcal{D}_{M-1} \subset \mathcal{D}_M$, property (8.4)

implies that $|B_\infty^H(s) - B_\infty^H(s_1)| \leq (b-1)b^{-MH'}$ and $|B_\infty^H(t) - B_\infty^H(t_1)| \leq (b-1)b^{-MH'}$. Moreover, since (8.5) holds for $m = M-1$ one has $|B_\infty^H(t_1) - B_\infty^H(s_1)| \leq 2(b-1) \sum_{j=n+1}^{M-1} b^{-H'j}$. This is enough to get (8.5) for $m = M$.

Property (8.5) being established for all $n \geq n_0$, taking $t, s \in \mathcal{D}$ such that $0 < |t-s| < b^{-n_0}$ and n the integer such that $b^{-(n+1)} \leq |t-s| < b^{-n}$, since both t and s belong to $\bigcup_{p \geq n} \mathcal{D}_p$ we deduce from (8.5) that

$$|B_\infty^H(t) - B_\infty^H(s)| \leq 2(b-1) \sum_{j=n+1}^{\infty} b^{-H'j} \leq \frac{2(b-1)}{1-b^{-H'}} |t-s|^{H'}.$$

This is enough to construct on $[0, 1]$ a unique continuous extension of B_∞^H . As a consequence of what precedes, this extension belongs to $C^{H'}([0, 1])$ for all $H' < H$.

(4) We need the following lemma which describes the asymptotic behavior of the characteristic function of $B_\infty^H(1)$. This lemma will be also useful in finding a lower bound for the Hausdorff dimension of the graph of B^H .

Lemma 8.2. *Let φ stand for the characteristic function of $B_\infty^H(1)$. There exists $\rho \in (0, 1)$ such that $\varphi(t) = O(\rho^{|t|^{1/H}})$ ($|t| \rightarrow \infty$). In particular, the probability distribution of $B_\infty^H(1)$ possesses an infinitely differentiable density.*

Proof. Since $\mathbb{E}(B_\infty^H(1)) = 1$, the probability distribution of $B_\infty^H(1)$ is not concentrated at 0 and thus for every $\eta > 0$ there exists $\alpha \in (0, \eta)$ and $\gamma < 1$ such that $\sup_{t, |t| \in [\alpha, b^H \alpha]} |\varphi(t)| \leq \gamma$.

Now, using the fact that

$$\varphi(t) = \left[p_{b,H}^+ \varphi(b^{-H}t) + p_{b,H}^- \varphi(-b^{-H}t) \right]^b,$$

one obtains by induction that

$$\sup_{t, |t| \in [b^{kH} \alpha, b^{(k+1)H} \alpha]} |\varphi(t)| \leq \gamma^{b^k} \quad (\forall k \geq 0).$$

Since $|t|^{1/H} \leq b \alpha^{1/H} b^k$ for $|t| \in [b^{kH} \alpha, b^{(k+1)H} \alpha]$, the conclusion follows with $\rho = \gamma^{1/b \alpha^{1/H}}$.

The rate of decay of φ at ∞ yields the conclusion regarding the probability distribution of $B_\infty^H(1)$. \square

It follows from Lemma 8.2 that $\mathbb{E}(|B_\infty^H(1)|^{-\gamma}) < \infty$ for all $\gamma \in (0, 1)$. This will be used with $\gamma = 1/2$ in what follows.

We next use an approach similar to that used for the study of the pointwise Hölder exponents of Brownian motion [8, 13].

Let $\varepsilon > 0$. We show that the subset \mathcal{O} of Ω of points ω such that the corresponding path B_∞^H possesses points at which the pointwise Hölder exponent is at least $H + \varepsilon$ is included in a set of null probability.

We fix an integer $K > 4/\varepsilon$ and denote by n_K the smallest integer n such that $Kb^{-n} \leq 1$. For $t \in [0, 1]$ and $n \geq n_K$, consider $S_n^K(t)$ a subset of $[0, 1]$ made of $K+1$ consecutive b -adic numbers of generation n such that $t \in [\min S_n^K(t), \max S_n^K(t)]$. Also denote by $\mathcal{S}_n^K(t)$ the set of K consecutive b -adic intervals delimited by the elements of $S_n^K(t)$. If the pointwise Hölder exponent at t is larger than or equal to $H + \varepsilon$ then for n large enough one has necessarily $\sup_{s \in S_n^K(t)} |B_\infty^H(s) - B_\infty^H(t)| \leq (Kb^{-n})^{H+\varepsilon/2}$, so that $\sup_{I \in \mathcal{S}_n^K(t)} |\Delta B_\infty^H(I)| \leq 2(Kb^{-n})^{H+\varepsilon/2}$.

Now let \mathbf{S}_n^K be the set made of all K -uple of consecutive b -adic intervals of generation n , and if $S \in \mathbf{S}_n^K$, denote the event $\{\sup_{I \in S} |\Delta B_\infty^H(I)| \leq 2(Kb^{-n})^{H+\varepsilon/2}\}$ by E_S . The previous lines show that

$$\mathcal{O} \subset \mathcal{O}' = \bigcap_{n \geq n_K} \bigcup_{p \geq n} \bigcup_{S \in \mathbf{S}_p^K} E_S.$$

By construction, if $S \in \mathbf{S}_p^K$, $(|\Delta B_\infty^H(I)|)_{I \in S}$ is equal to $(b^{-pH}Y_I)_{I \in S}$, where the K random variables Y_I are mutually independent and identically distributed with $|B_\infty^H(1)|$. Consequently, $\mathbb{P}(E_S)$ depends only on K and p and

$$\mathbb{P}(E_S) \leq \left[\mathbb{P}(|B_\infty^H| \leq 2(Kb^{-p})^{\varepsilon/2}) \right]^K \leq \sqrt{2}^K K^{K\varepsilon/4} b^{-pK\varepsilon/4} \left[\mathbb{E}(|B_\infty^H(1)|^{-1/2}) \right]^K.$$

Since the cardinality of \mathbf{S}_p^K is less than b^p , this yields $\mathbb{P}(\bigcup_{S \in \mathbf{S}_p^K} E_S) \leq Cb^p b^{-pK\varepsilon/4}$, with $C = \sqrt{2}^K K^{K\varepsilon/4} [\mathbb{E}(|B_\infty^H(1)|^{-1/2})]^K$. Due to our choice for K , this implies that the series $\sum \mathbb{P}(\bigcup_{S \in \mathbf{S}_p^K} E_S)$ converges and $\mathbb{P}(\mathcal{O}') = 0$. \square

The next proposition makes it possible to conclude that the random sequence of functions $(B_n^H)_{n \geq 1}$ converges almost surely uniformly to the function B_∞^H constructed previously. The same kind of approach is used to establish the convergence of continuous function-valued martingales related to multiplicative processes on a homogeneous or Galton Watson tree in [11, 3, 1], but the context in the mentioned papers is rather different from the present one because the martingales $M_n(s)$ considered there take the form $\sum_{w \in \Sigma_n} \prod_{k=1}^n W(w|k)(s)$ where the random weights $W(w)(s)$ depend smoothly on the parameter s belonging to some open subset of \mathbb{R}^d independently of Σ (and more generally a super-critical Galton-Watson tree), while for $B_n^H(s)$ the parameter s is a generic point in Σ (identified with $[0, 1]$).

Proposition 8.2. *One has $\mathbb{E}(\|B_\infty^H\|_\infty) < \infty$. Consequently, with probability 1, B_n^H converges uniformly to B_∞^H .*

Proof. In fact we are going to prove that $\mathbb{E}(\|B_\infty^H\|_\infty^2) < \infty$. Define

$$\tilde{Z}_n = \sup_{t \in \bigcup_{p \geq 1} \mathcal{D}_p} |B_n^H(t)|, \quad \tilde{Z}_n(k) = \sup_{t \in \bigcup_{p \geq 1} \mathcal{D}_p} |B_n^H(k)(t)|, \quad 0 \leq k \leq b-1.$$

Due to (2.1) one has

$$\tilde{Z}_n \leq \max_{0 \leq k \leq b-1} b^{-H} \tilde{Z}_{n-1}(k) + |B_n^H(kb^{-1})| \leq \sum_{k=0}^{b-1} b^{-H} \tilde{Z}_{n-1}(k) + |B_n^H(kb^{-1})|$$

Thus

$$\begin{aligned} \mathbb{E}(\tilde{Z}_n^2) &\leq \sum_{k=0}^{b-1} \mathbb{E} \left(b^{-2H} \tilde{Z}_{n-1}(k)^2 + 2\tilde{Z}_{n-1}(k) |B_n^H(kb^{-1})| + |B_n^H(kb^{-1})|^2 \right) \\ &= b^{1-2H} \mathbb{E}(\tilde{Z}_{n-1}^2) + 2\mathbb{E} \left(\sum_{k=0}^{b-1} \tilde{Z}_{n-1}(k) |B_n^H(kb^{-1})| \right) + \sum_{k=0}^{b-1} \mathbb{E}(|B_n^H(kb^{-1})|^2) \\ &\leq b^{1-2H} \mathbb{E}(\tilde{Z}_{n-1}^2) + 2\mathbb{E}(\tilde{Z}_{n-1}^2)^{1/2} \sum_{k=0}^{b-1} \|B_n^H(kb^{-1})\|_2 + \sum_{k=0}^{b-1} \|B_n^H(kb^{-1})\|_2^2. \end{aligned}$$

Now we use the fact that the sequence $\sup_{0 \leq k \leq b-1} \|B_n^H(kb^{-1})\|_2$ is bounded due to the proof of Proposition 8.1. Thus there exists $C > 0$ such that for all $n \geq 1$

$$(8.6) \quad \mathbb{E}(\tilde{Z}_n^2) \leq f(\mathbb{E}(\tilde{Z}_{n-1}^2)), \text{ with } f(x) = b^{1-2H}x + C\sqrt{x} + C.$$

Since $b^{1-2H} < 1$, there exists $x_0 > 0$ such that $f(x) < x$ for all $x > x_0$. This remark together with (8.6) implies that $\mathbb{E}(\tilde{Z}_n^2) \leq \max(x_0, f(\mathbb{E}(\tilde{Z}_1^2)))$ for all $n \geq 2$.

To conclude, we use Proposition V-2-6 in [21] which ensures that since $\mathcal{C}([0, 1])$ is a complete separable Banach space and $\|B_\infty^H\|_{L^1} < \infty$, B_∞^H is the almost sure limit of $\tilde{B}_n^H = \mathbb{E}(B_\infty^H | \sigma(\epsilon(w) : |w| \leq n))$. Furthermore, given $n \geq 1$ and $w \in \Sigma_n$, one can show by induction on $p \geq 0$ that, with probability 1, $\tilde{B}_n^H(t_{w \cdot u}) = B_n^H(t_{w \cdot u})$ for all $u \in \Sigma_p$. This implies that $\tilde{B}_n^H = B_n^H$ almost surely since these functions coincide over \mathcal{D} . \square

Proposition 8.3. *Let $H \in (1/2, 1]$. With probability 1, the Hausdorff and box dimensions of the graph of B_H are equal to $2 - H$.*

We shall need an additional notation. If $w \in \Sigma^*$ and $J = \pi([w])$ then we define $\epsilon(J) = \prod_{k=1}^{|w|} \epsilon(w|k)$.

Proof. Let us denote by Γ_H the graph $\{(t, B_H(t)) : t \in [0, 1]\}$ of B_H .

At first, the fact that $2 - H$ is an upper bound for the box dimension of the graph of B_H comes from the fact that $B_H \in C^{H'}([0, 1])$ for all $H' < H$ (see [9] Ch. 11).

To find the sharp lower bound $2 - H$ for the Hausdorff dimension of Γ_H , we use the method consisting in showing that with probability 1, the measure on this graph obtained as the image of the Lebesgue measure restricted to $[0, 1]$ by the mapping $t \mapsto (t, B_H(t))$ has a finite energy with respect to the Riesz Kernel $u \in \mathbb{R}^2 \setminus \{0\} \mapsto \|u\|^{-\gamma}$ for all $\gamma < 2 - H$ (see [9] Ch. 11 for more details). This property holds if we show that for all $\gamma < 2 - H$

$$\int_{[0,1]^2} \mathbb{E} \left(\frac{1}{\sqrt{|t-s|^2 + |B_H(t) - B_H(s)|^2}^\gamma} \right) dt ds < \infty.$$

If I is a closed subinterval of $[0, 1]$, we denote by $\mathcal{G}(I)$ the set of closed b -adic intervals of maximal length included in I , and then $m_I = \min \bigcup_{J \in \mathcal{G}(I)} J$ and $M_I = \max \bigcup_{J \in \mathcal{G}(I)} J$.

Let $0 < s < t < 1$ be two non b -adic numbers. We define two sequences $(s_p)_{p \geq 0}$ and $(t_p)_{p \geq 0}$ as follows. Let $s_0 = m_{[s,t]}$ and $t_0 = M_{[s,t]}$. Then let define inductively $(s_p)_{p \geq 1}$ and $(t_p)_{p \geq 1}$ as follows: $s_p = m_{[s, s_{p-1}]}$ and $t_p = M_{[t_{p-1}, t]}$. Let us denote by \mathcal{C} the collection of intervals containing $[s_0, t_0]$ and the intervals $[s_p, s_{p-1}]$ and $[t_{p-1}, t_p]$, $p \geq 1$. Every interval $I \in \mathcal{C}$ is the union of at most $b - 1$ intervals of the same generation n_I , the elements of $\mathcal{G}(I)$, and we have $\Delta B_H(I) = \sum_{J \in \mathcal{G}(I)} \epsilon(J) b^{-n_I H} Y_J$.

By construction, we have $\min_{I \in \mathcal{C}} n_I = n_{[s_0, t_0]}$ and $(t - s)/b \leq b^{-n_{[s_0, t_0]}} \leq (t - s)$.

Also, all the random variables Y_I are mutually independent and independent of $\mathcal{T}_{\mathcal{C}} = \sigma(\epsilon(J) : J \in \mathcal{G}(I), I \in \mathcal{C})$.

Now, we write

$$B_H(t) - B_H(s) = b^{-n_{[s_0, t_0]}H} \left(\sum_{J \in \mathcal{G}([s_0, t_0])} \epsilon(J) Y_J + Z(s, s_0) + Z(t_0, t) \right),$$

where

$$\begin{cases} Z(s, s_0) = \lim_{p \rightarrow \infty} \sum_{0 \leq k \leq p} b^{(n_{[s_0, t_0]} - n_{[s_{k+1}, s_k]})H} \sum_{J \in \mathcal{G}([s_{k+1}, s_k])} \epsilon(J) Y_J \\ Z(t_0, t) = \lim_{p \rightarrow \infty} \sum_{0 \leq k \leq p} b^{(n_{[s_0, t_0]} - n_{[t_k, t_{k+1}])H} \sum_{J \in \mathcal{G}([t_k, t_{k+1}])} \epsilon(J) Y_J. \end{cases}$$

Let $Z(t, s) = \sum_{J \in \mathcal{G}([s_0, t_0])} \epsilon(J) Y_J + Z(s, s_0) + Z(t_0, t)$ and fix $J_0 \in \mathcal{G}([s_0, t_0])$. Conditionally on \mathcal{T}_C , $Z(t, s)$ is the sum of $\pm Y(J_0)$ plus a random variable U independent of $Y(J_0)$. Consequently, the probability distribution of $Z(t, s)$ conditionally on \mathcal{T}_C possesses a density $f_{t,s}$ and $\|\widehat{f_{t,s}}\|_{L^1} \leq \|\varphi\|_{L^1}$, where φ is the characteristic function of $Y(J_0)$ studied in Lemma 8.2.

Thus, for $\gamma < 2 - H$ we have

$$\begin{aligned} \mathbb{E} \left(\frac{1}{\sqrt{|t-s|^2 + |B_H(t) - B_H(s)|^2}^\gamma} \middle| \mathcal{T}_C \right) &= \int_{\mathbb{R}} \frac{f_{t,s}(u)}{\sqrt{|t-s|^2 + b^{-2n_{[s_0, t_0]}H} u^2}^\gamma} du \\ &\leq \int_{\mathbb{R}} \frac{f_{t,s}(u)}{\sqrt{|t-s|^2 + b^{-2H}(t-s)^{2H} u^2}^\gamma} du \\ &= |t-s|^{1-H-\gamma} \int_{\mathbb{R}} \frac{f_{t,s}(|t-s|^{1-H} v)}{\sqrt{1 + b^{-2H} v^2}^\gamma} dv. \end{aligned}$$

The function $f_{t,s}$ is bounded independently of t, s and \mathcal{T}_C since it is bounded by $\|\widehat{f_{t,s}}\|_{L^1}$ and we just saw that this number is bounded by $\|\varphi\|_{L^1}$. It follows that

$$\mathbb{E} \left(\frac{1}{\sqrt{|t-s|^2 + |B_H(t) - B_H(s)|^2}^\gamma} \right) \leq \|\varphi\|_{L^1} |t-s|^{1-H-\gamma} \int_{\mathbb{R}} \frac{dv}{\sqrt{1 + b^{-2H} v^2}^\gamma}.$$

This yields the conclusion. \square

9. PROOF OF THEOREMS 6.1 AND 6.2 CONCERNING FUNCTIONAL CLT WHEN $1/2 < H < 1$

Proofs of Theorem 6.1 and 6.2. We proceed in three steps as for Theorem 3.1.

Let $a_n^H = \sigma_H b^{n(1/2-H)}$. For $w \in \Sigma^*$ and $n \geq 1$, let $Y_n(w) = (B^H(w) - B_n^H(w))(1/a_n^H)$, and simply denote $Y_n(\emptyset)$ by Y_n . By construction, $\mathcal{L}(Y_n(w)) = \mathcal{L}(Y_n)$. Also, for $n \geq 1$ let $\mathcal{X}_n^H = B^H - B_n^H$.

Step 1: We leave the reader verify that

$$(9.1) \quad Y_n = \frac{1}{\sqrt{b}} \sum_{k=0}^{b-1} \epsilon(k) Y_{n-1}(k),$$

where $Y_{n-1}(k) \sim Y_{n-1}$, and the $Y_{n-1}(k)$'s are centered, mutually independent, and independent of the $\epsilon(k)$'s.

It is then straightforward to show by induction that properties (1), (2) and (3) of the proof of Proposition 7.1 hold (with the new sequence (Y_n) considered in the present proof).

Thus $X_n^H(1) = \mathcal{X}_n^H(1)$ converges weakly to $\mathcal{N}(0, 1)$.

Step 2: Let $(W(t))_{t \in [0,1]}$ be a standard Brownian motion. For every $p \geq 1$, the probability distributions $\mathcal{L}\left((X_{p+n}^H(t))_{t \in \mathcal{D}_p}\right) = \mathcal{L}\left((\mathcal{X}_{p+n}^H(t))_{t \in \mathcal{D}_p}\right)$ converge to $\mathcal{L}((W(t))_{t \in \mathcal{D}_p})$ as $n \rightarrow \infty$. This is obtained by following the same approach as in the proof of Proposition 7.2, as well as the step 1 and the fact that if $w \in \Sigma_p$, for all $n \geq p$ we have

$$(9.2) \quad \Delta \mathcal{X}_n^H(I_w) = \Delta X_n^H(I_w) = b^{-p/2} Y_{n-p}(w).$$

Step 3: To see that the sequences $(\mathcal{L}(X_n^H))_{n \geq 1}$ and $(\mathcal{L}(\mathcal{X}_n^H))_{n \geq 1}$ are tight, we follow the same approach as in Proposition 7.3.

For $n \geq 2$ and $p \geq 1$, we notice that:

If $1 \leq p \leq n$ and $w \in \Sigma_p$ then we have $\Delta \mathcal{X}_n^H(I_w) = \Delta X_n^H(I_w) = b^{-p/2} Y_{n-p}(w)$ (this is (9.2)).

If $p > n$ and $w \in \Sigma_p$ then we have

$$\begin{aligned} |\Delta X_n^H(I_w)| &= |\Delta X_n^H(I_{w|n})| b^{n-p} = \sigma_H^{-1} b^{n/2-p} |B^H(w|n)(1) - 1| \\ &\leq \sigma_H^{-1} b^{-p/2} |B^H(w|n)(1) - 1|. \end{aligned}$$

and

$$\begin{aligned} |\Delta \mathcal{X}_n^H(I_w)| &= (a_n^H)^{-1} |\Delta B^H(I_w) - \Delta B_n^H(I_w)| \\ &= \sigma_H^{-1} b^{n(H-1/2)} \left| b^{-pH} B^H(w)(1) \prod_{k=1}^p \epsilon(w|k) - b^{-nH} b^{n-p} \prod_{k=1}^n \epsilon(w|k) \right| \\ &= \sigma_H^{-1} b^{-p/2} \left| b^{(p-n)(1/2-H)} B^H(w)(1) \prod_{k=n+1}^p \epsilon(w|k) - b^{-(p-n)/2} \right|. \end{aligned}$$

Now, since by Lemma 8.1 and the step 1 the sequences $Y_n(w)$, $|B^H(w|n)(1) - 1|$ and $|b^{(p-n)(1/2-H)} B^H(w)(1) \prod_{k=n+1}^p \epsilon(w|k) - b^{-(p-n)/2}|$ are bounded in L^q for all $q \geq 1$ independently of w , we can deduce (by using an approach similar to that used in the proof of Proposition 7.3) from the previous estimates of $|\Delta \mathcal{X}_n^H(I_w)|$ and $|\Delta X_n^H(I_w)|$ that for every $H' < 1/2$, there exists a positive sequence $(\beta_p)_{p \geq 1}$ such that $\sum_{p \geq 1} \beta_p < \infty$ and

$$\sup_{n \geq 2} \mathbb{P}(\exists w \in \Sigma_p : |\Delta X_n^H(I_w)| > b^{-pH'}) + \sup_{n \geq 2} \mathbb{P}(\exists w \in \Sigma_p : |\Delta \mathcal{X}_n^H(I_w)| > b^{-pH'}) \leq \beta_p.$$

In view of the proof of Proposition 7.3, this is enough to establish the desired tightness.

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