Signals and Systems
Lecture 13

Laplace Transforms

April 28, 2008

Today's Topics

1. Definition of the Laplace transform
2. Regions of convergence of Laplace Transforms

Take Away

The Laplace transform has many of the same properties as Fourier transforms but there are some important differences as well.

Required Reading

O&W-9.0, 9.1(except Example 9.2), 9.2, 9.9

Although the Fourier transform is an extremely useful tool for analyzing many kinds of systems it has some shortcomings that can be overcome, in many ways, by the Laplace Transform. In particular, the Fourier transform is not very useful for studying the stability of systems because, in studying instabilities, it is often necessary to deal with signals that diverge in time. We know that the Fourier integral does not converge for signals that diverge because such signals are not absolutely integrable. As we will see, the Laplace transform alleviates this problem in most situations of interest, thereby enabling us to study divergent signals and unstable systems.

Recall that the response of a LTI system, with impulse response *h(t),* to a complex exponential input is

$$y(t) = H(s)\, e^{st}$$

where the complex variable s is defined as having a real part σ and imaginary part ω

$$s = \sigma + j\omega$$

and *H(s)* is the integration of the system impulse response times the input complex exponential.

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

This integral is defined as the Laplace transform of *h(t)*.

The relationship of the Laplace transform to the Fourier transform is readily apparent if we restrict s to be purely imaginary (i.e., $s = j\omega$) . Then the Laplace transform of some function of time *x(t)* is identical to the Fourier transform of *x(t)*. If *X(s)* is the Laplace transform of *x(t)* then if *x(t)* has a Fourier transform

$$X(s)\Big|_{s=j\omega} = \mathcal{F}(x(t))$$

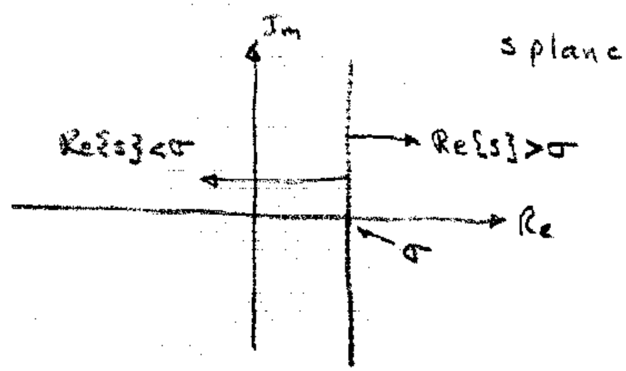
Alternatively, we can write the Laplace transform of *x(t)* as

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

or

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t} \right] e^{-j\omega t}\, dt$$

and recognize the right hand side of this equation as the Fourier transform of $x(t)e^{-\sigma t}$. In other words, the Laplace transform can be thought of as the Fourier transform of a signal that has been modified by multiplying it by $e^{-\sigma t}$. Depending upon the value of σ, which is the real part of *s,* the signal *x(t)* can be multiplied by a decaying or expanding exponential. Also, the value of σ is indicative of sets of locations in the complex plane, as illustrated in the following figure.



We will tacitly assume that $x(t)e^{-\sigma t}$ is of bounded variation and has only a finite number of discontinuities, which are two of the three Dirichlet conditions that assure its Fourier transform.

If, in addition, $x(t)e^{-\sigma t}$ is absolutely integrable then the Fourier transform of $x(t)e^{-\sigma t}$ exists. But the Fourier transform of $x(t)e^{-\sigma t}$ is the Laplace transform of $x(t)$, so the condition

$$\int_{-\infty}^{\infty} \left| e^{-\sigma t} x(t) \right| dt < \infty$$

assures the convergence of the Laplace transform of *x(t)*. The region in the "*s*" plane where this infinite integral converges is called the region of convergence (ROC).

<u>Example</u>

Suppose *x(t)* is

$$x(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases} \qquad a > 0$$

which is a decaying exponential for *t>0*. The Fourier transform exists because all Dirichlet conditions are satisfied and

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \frac{1}{(j\omega + a)}$$

The Laplace transform of *x(t)* is

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{0}^{\infty} e^{-at} e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt = \int_{0}^{\infty} e^{-(\sigma+a)t} e^{-j\omega t} dt$$

but this is just the Fourier transform of $e^{-(\sigma+a)t}$ so

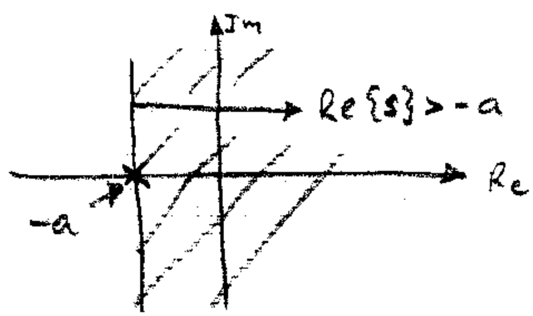
$$X(s) = X(\sigma + j\omega) = \frac{1}{(\sigma+a)+j\omega} \qquad \sigma + a > 0$$

or, equivalently, since

$$\sigma = Re\{s\}$$

then the Laplace transform of *x(t)* is

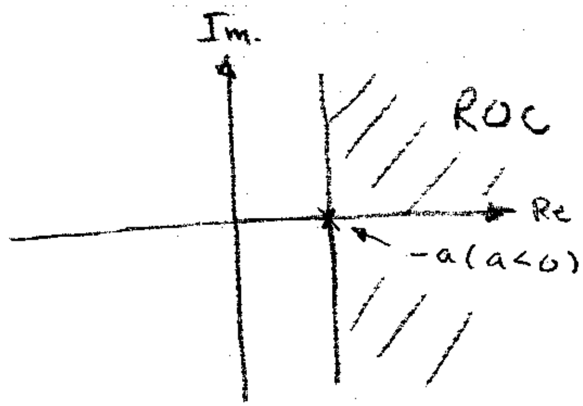
$$X(s) = \frac{1}{s+a} \qquad Re\{s\} > -a$$

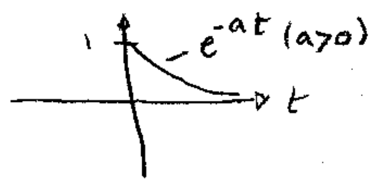


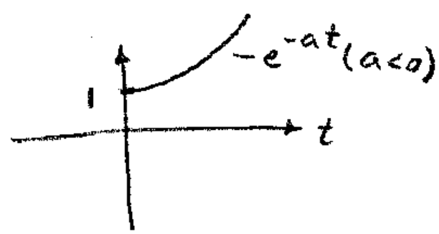
The point in the "s" plane where *X(s)* goes to infinity, namely where s=-a, is called a pole of *X(s)*. This point is typically indicated by an "x", as shown in the figure. The region to the right of *–a* in the *s* plane is the ROC.

Recall that the Fourier transform will not exist for signals that are not absolutely integrable. For the Laplace transform this requirement is now altered by the condition on the region of convergence, namely that the real part of *s* must be in the ROC.

In the example above the ROC is the region in the complex plain for which the real part of *s* is greater than $-a$, as shown in the diagram. The Fourier transform for $x(t)$ only converges for $a>0$ because if $a<0$ then $x(t)$ diverges as time goes to infinity and is not absolutely integrable. However, the Laplace transform of $x(t)$ will converge for <u>any</u> value of a as long as the real part of s is greater than $-a$, which is equivalent to requiring that $\sigma > -a$)







<u>Thus, the Laplace transform allows us to deal with signals that diverge in time and hence we will be able to analyze unstable systems</u>.

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Example -Satellite Attitude Dynamics

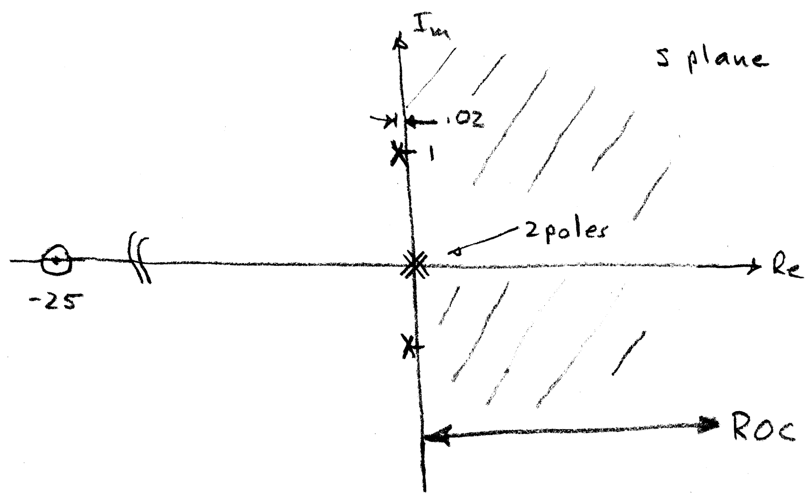
In this example will use the transfer function for the satellite dynamics that we derived in Lecture 11

$$H(s) = H(j\omega)\Big|_{s=j\omega} = \frac{.036(s+25)}{s^2(s^2+.04s+1)}$$

The roots of the denominator quadratic function are

$$s_p = \frac{-.04 \pm \sqrt{(.04)^2-4}}{2} = -.02 \pm 1.0j$$

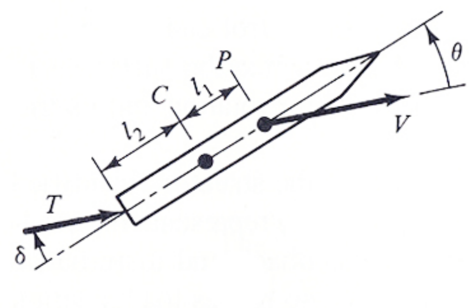
and if we plot the roots in the complex plane



The ROC is the entire right half plane. Note that the numerator term determines a point in the plane where *H(s)* goes to zero, namely at *s=-25*. This point is called a zero and is indicated by an "o" in the plane.

Example-Unstable Rocket

Commonly rocket vehicles are inherently unstable. For example, consider the greatly simplified rocket vehicle shown in the diagram.



(a)

The point C is the center of mass, the point P is the center of aerodynamic pressure due to air flow over the body, T is the thrust of the rocket engine that acts at the angle δ on the base of the vehicle, V is the relative velocity of airflow and θ is the angle of attack. We assume that θ is a small angle so the lift force on the vehicle is linearly proportional to θ, and is assumed to act orthogonal to the body centerline at the point P.

The lift force is

$$L = C_n \theta$$

where C_n is a normal force coefficient that embodies all of the lift effects such as dynamic pressure, equivalent area, etc. Assuming that δ is also a small angle the component of thrust that acts orthogonal to the body centerline is

$$T_\perp = T \sin \delta \cong T\delta$$

Thus, if we sum the moments about the center of mass we can apply Newton's equation to obtain

$$J\ddot{\theta} = C_n \ell_1 \theta + T \ell_2 \delta$$

where J is the moment of inertia about the center of mass.

If we take the Fourier transform of all terms in this equation we obtain

$$J(j\omega)^2\Theta(j\omega) = C_n \ell_1 \Theta(j\omega) + T\ell_2 \Delta(j\omega)$$

where $\Theta(j\omega)$ is the Fourier transform of the angle of attack and $\Delta(j\omega)$ is the Fourier transform of the thrust angle. We can immediately obtain the frequency response of the system as

$$H(j\omega) = \frac{\Theta(j\omega)}{\Delta(j\omega)} = \frac{(T\ell_2/J)}{(j\omega)^2 - (C_n\ell_1/J)}$$

and hence the input/output Laplace transform is simply obtained by substituting s for $j\omega$

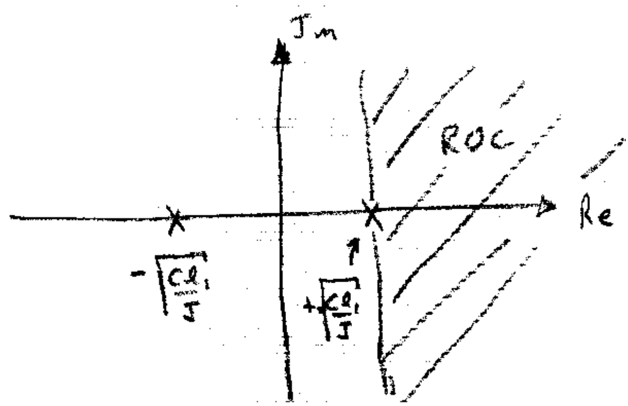
$$H(s) = \frac{(T\ell_2/J)}{s^2 - (C_n\ell_1/J)}$$

and we call this the <u>system transfer function</u> because it relates the Laplace transform of the output to the Laplace transform of the input.

This system has two poles at

$$s_p = \pm\sqrt{\frac{C_n\ell_1}{J}}$$

and in the s plane



Thus there is a pole in the right half plane and the ROC is to the right of that pole. In fact, as we will see in future lectures, the rocket is unstable so its impulse response diverges and is not absolutely integrable. Hence the Fourier transform of the impulse response does not converge. However, the Laplace transform does exist with the ROC as indicated in the diagram.

In what follows we will restrict our inquiry to signals for time greater than or equal to zero and to causal systems. The assumption that signals are zero for all negative time implies that there is some point in time, that we will call the initial or zero time, before which we are not interested in the behavior of the signal. Similarly, a causal system will only respond to inputs in real time. It cannot predict the future! Hence the impulse response of a causal system must be zero for time less than zero. Hence from now on we will require that

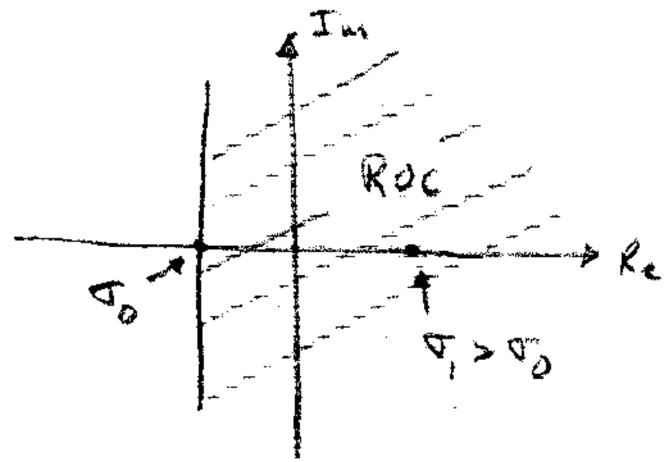
$$\left.\begin{aligned} x(t) &= 0 \\ h(t) &= 0 \end{aligned}\right\} \quad t < 0$$

Since we are restricting our exploration of Laplace transforms to signals and impulse responses that have zero values for time less than zero our definition of the Laplace transform is in fact the unilateral Laplace transform

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

We call it the unilateral Laplace transform to distinguish it from the bilateral Laplace transform which includes signals for time less than zero and integrates from $-\infty$ to $+\infty$. The unilateral Laplace transform is the most common form and is usually simply called the Laplace transform, which is what we will call it. This transform has a number of properties that are explained in the text. We will only need three for what we will be doing. These properties are-

Property A If the line Re{s}=σ_0 is in the ROC of the Laplace transform of *x(t)* then all points in the complex plane for which $\sigma > \sigma_0$ are also in the ROC.

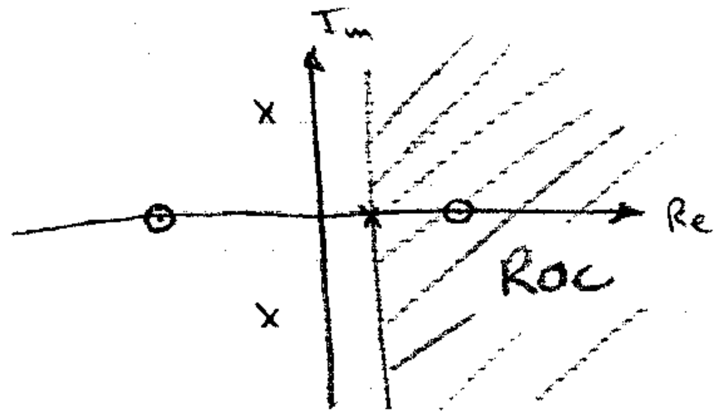


This property follows from the fact that if the Laplace transform of *x(t)* converges for $\sigma = \sigma_0$ then it also converges for any $\sigma_1 > \sigma_0$

Property B For rational Laplace transforms the ROC does not contain any poles. This property simply recognizes that the Laplace transform goes to infinity at a pole so the Laplace transform integral will not converge at that point and hence it cannot be in the ROC.

Property C If the Laplace transform of *x(t)* is rational then the ROC is the region to the right of the right most pole in the complex plane.

This property is a direct consequence of the fact that a rational Laplace transform contains a finite number of real and complex conjugate poles. As we showed in two earlier examples the ROC for real or complex conjugate poles is the entire region to the right of the poles. This fact, along with Properties A and B assure Property C.



The Inverse Laplace Transform

Earlier we discussed the interpretation of the Laplace transform of a function as the Fourier transform of that function, multiplied by a real exponential. In particular, if $s = \sigma + j\omega$ then the Laplace tranform of $x(t)$ is

$$X(s) = X(\sigma + j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]\, e^{-j\omega t}\, dt$$

for $s = \sigma + j\omega$ in the ROC. We can invert this relationship using the inverse Fourier transform. Thus

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

and multiplying through by $e^{\sigma t}$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}\,d\omega$$

or, if we change the variable of integration to $s = \sigma + j\omega$ and recognize that σ is a constant in this context, then

$$x(t) = \frac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

Although we will need this equation later, we will see that this formula is not very useful for actually determining inverse Laplace transforms and we will usually employ other methods.