Time-Localized Wavelet Multiple Regression and Correlation: Eurozone stock markets across scales and time

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Abstract

This paper extends wavelet methodology to handle comovement dynamics of multivariate time series via moving weighted regression on wavelet coefficients. The concept of wavelet local multiple correlation is used to produce one single set of multiscale correlations along time, in contrast with the large number of wavelet correlation maps that need to be compared when using standard pairwise wavelet correlations with rolling windows. Also, the spectral properties of weight functions are investigated and it is argued that some common time windows, such as the usual rectangular rolling window, are not satisfactory on these grounds.

The method is illustrated with a multiscale analysis of the comovements of Eurozone stock markets during this century. It is shown how the evolution of the correlation structure in these markets has been far from homogeneous both along time and across timescales featuring an acute divide across timescales at about the quarterly scale. At longer scales, evidence from the long-term correlation structure can be interpreted as stable perfect integration among Euro stock markets. On the other hand, at intramonth and intraweek scales, the short-term correlation structure has been clearly evolving along time, experiencing a sharp increase during financial crises which may be interpreted as evidence of financial 'contagion'.

Keywords: Comovement dynamics, Euro zone, local regression, multiscale analysis, multivariate time series, non-stationary time series, stock markets, wavelet transform, weighted least squares.

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1. Introduction

One important aspect in the analysis of economic and financial time series is the study of the degree and behavior of their comovements at different periods and frequencies. Whilst traditional time series analysis (TSA) is mostly based on cross-correlation functions

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in the time domain, that is, on a observation-by-observation basis, Fourier analysis allows for the visualization of the data in the frequency domain, that is, on a frequency-by-frequency basis. In other words, traditional TSA provides full time resolution of such relationships but does not expose frequency information while Fourier analysis has full frequency resolution but does not preserve information in time. These types of analysis implicitly assume stationarity, possibly after differences, as the main characteristic of the time series under study (Hamilton, 1994, p.435).

The more recent wavelet analysis emerges as a compromise between both approaches, with partial resolution in both time and frequency domains. Thus, the continuous wavelet transform (CWT) allows for a visualization of the spectral features of the time series and their comovements but as a function of both time and scale (frequency), separating their different periodic components as they evolve over time (Percival and Walden, 2000; Aguiar-Conraria and Soares, 2014).

The CWT is highly redundant on both time and scale dimensions. The latter not being a desirable feature for regression/correlation decomposition over timescales. On the other hand, the discrete wavelet transform (DWT) selects a minimal subsample of time-frequency values from the CWT without loosing any information present in the original data. This is an important feature for some engineering applications like signal and image compression, but for most economic applications time redundancy is somehow desirable as long as it allows for data features to be properly aligned and compared across all scales/frequencies. In this sense, the maximal overlap discrete wavelet transform (MODWT) is the most popular wavelet transform as it is redundant in the time dimension but non-redundant in the scale/frequency dimension. MODWT is known to have several important advantages, including energy preservation which is particularly important for the main objective of this paper (Percival and Mofjeld, 1997; Percival and Walden, 2000; Gençay et al., 2002, p.135).

MODWT based wavelet multiple correlations (WMC) and cross-correlations (WMCC) were introduced by Fernández-Macho (2012). These statistics distribute among the different timescales the overall statistical relationship that might exist between several time series and they have received some attention in the recent literature (see *e.g.* Addison, 2017; Aguiar-Conraria and Soares, 2014; Benhmad, 2013; Berger, 2015; Chakrabarty et al., 2015; Huang et al., 2015; Khalfaoui et al., 2015; Saâdaoui et al., 2017; Shahzad et al., 2017; Sousa et al., 2014, among others). However, as with standard bivariate wavelet correlations, it is implicitly assumed that the time series follow difference stationary processes and, therefore, that there exists a sufficiently long wavelet filter that eliminates this type of nonstationarity from the data. In consequence, only one single global correlation per scale needs be produced.

This notwithstanding, with economic and financial time series, such data features are probably non-stationary in nature and a regression/correlation analysis must be able to handle and visualize a changing structure that evolves along time. Therefore, the present paper generalizes the previous global WMC to a local multiple regression framework where comovement dynamics across the different scales/frequencies can be analyzed along time by using weighted or windowed wavelet coefficients.

The proposed method is justified on several grounds. First of all, the alternative of combining standard bivariate wavelet correlation analysis with rolling time windows needs to calculate, plot and compare a large number of wavelet correlation graphs that now would require an additional time dimension (Ranta, 2010; Dajcman et al., 2012; Ranta, 2013; Benhmad, 2013, *etc.*). More specifically, with n time series, a total of $n(n-1)/2$ wavelet correlation maps would be required each of dimension $J \times T$, where $J = \lfloor \log_2(T/(L-1)+1) \rfloor$ is the order of the wavelet transform, L is the wavelet filter length and T is the time series length.[1] Not surprisingly, when dealing with all possible pairwise comparisons in a multiscale context, the analyst may end up with a vast amount of potentially conflicting information that can be very difficult to process and lead to the typical experimentwise error rate inflation and the spurious detection of correlations at some wavelet scales (Fernández-Macho, 2012). In contrast, the proposed method based on local multiple regression, consists in one single set of multiscale relationships which can be expressed in a single scale-time correlation map. This is not only easier to handle and interpret but also may provide a better insight of the overall statistical comovement dynamics within the multivariate time series under scrutiny. Also, in the proposed method the number of feasible scales remains the same as in global wavelet analysis and does not depend on the length of the time window. Therefore, long wavelet filters and windows of long length or even with infinite support like the gaussian weight function can be used. Finally, as discussed in section 5, the spectral properties of the weight function need to be taken into account as, for example, the usual rectangular rolling window will not be satisfactory on these grounds.

All this will be illustrated with the application of the proposed Wavelet Local Multiple Correlation (WLMC) in the multiscale analysis of daily returns obtained from a set of eleven Eurozone stock markets during the 17-year period of the present century during which several financial and debt crises have occurred. In this relation, we may point out that correlation among European stock markets is a common measure of market integration in the economic and financial literature (see, *e.g.*, Fratzscher, 2002; Yang et al., 2003; Hardouvelis et al., 2006; Syllignakis, 2006; Bartram et al., 2007, and others). However, these studies do not usually take into account the fact that stock markets involve heterogenous agents that make decisions over different time horizons (Gençay et al., 2002, p.10, Gallegati, 2008), or that such comovement structure across different timescales may be evolving along time. On the other hand, the relatively large but not uncommon number of markets to be analyzed will render, as already mentioned, pairwise multiscale comparisons pointless in practice, which is the reason why this type of market analysis may find useful the wavelet local multiple regression tools proposed here.

The paper is organized as follows. Sections 2 and 3 set up the framework for the proposed wavelet local multiple regression tools. Section 4 extends such framework considering the decomposition of the time series structure across different timescales

[1] For example, the empirical analysis in Section 7 would need to handle a total of 55 [9×4542] wavelet correlation maps.

and defines the WLMC. Sample estimators of these quantities and their approximate confidence intervals based on their large sample theory are also provided for estimation and testing purposes. Section 5 discusses the spectral properties of common weight functions in local regression that can be used in practice and Section 6 presents an example with simulated breaks in the correlation structure across different timescales that illustrates the validity of the proposed statistics. Finally, Section 7 shows the results of an empirical application using Eurozone stock markets and Section 8 summarizes the main conclusions.

2. Outline of the method

The proposed method consists of the following steps: a) discrete wavelet transform (MODWT), Section 4; b) weight function or rolling time window, *i.e.* moving averages, Section 5; c) local least-squares regression and d) local multiple-correlation coefficients, Section 3; or a-b-d if only correlations are needed.

The use of rolling-window correlations in bivariate wavelet analysis has been proposed by Ranta (2010); Dajcman et al. (2012); Ranta (2013); Benhmad (2013) among others. However, many studies seem to reverse the above order to b-a-d so that they calculate wavelet correlations on a rectangular rolling window of data. In principle, since both a) and b) are linear filters, it would appear that reversing this order does not matter. Indeed, this would be the case if boundary conditions on the rolling window take into account surrounding values from the complete series, but this is not done. Instead, each subsample selected by the rolling window is taken as if it were a complete series with the usual standard periodic or reflecting boundary conditions and wavelet coefficients affected by such boundaries customarily rejected. This explains why these authors complain of running out of data points as the wavelet level increases and, trying to mitigate this boundary problems, they are forced to use a filter with the shortest length such as the Haar filter, which is not recommended due to its poor band-pass properties (Ranta, 2013, p.142).[2] Using the a-b-d order solves these problems without further ado.

3. Local regression

Let $X \in \mathbb{R}^n \times \mathbb{R}$ be a n-variate time series observed at times t=1…T, and let $X_{-i} = X \setminus \{x_i\}$ for some $x_i \in X$. We wish to obtain a linear function $f(X_{-i})$ that for a fixed $s \in [1, \ldots, T]$, minimizes a weighted sum of squared errors

$$S_s = \min_f \sum_t \theta(t-s) \left[f(X_{-i,t}) - x_{it} \right]^2,$$

where $\theta(x)$ is a given moving average weight function, such as those in Table 1, that depends on the time lag between observations X_t and X_s (*cf. e.g.* Nealen, 2004).

[2] Benhmad (2013) seems to apply the preferred a-b-d order. Nevertheless, he uses DWT instead of MODWT for the scale decomposition and, therefore, he also runs out of data points as the wavelet level increases.

Thus, the local weighted least squares approximation around s can be written as

$$f(X_{-i}) = Z_i \beta_s, \quad Z_i = X_{-i} - \overline{X}_{-i},$$

where the unknown coefficients β_s can be estimated as

$$\widehat{\beta}_s = \big[\sum_t \theta(t-s) Z_{it} Z'_{it}\big]^{-1} \sum_t \theta(t-s) Z_{it} x_{it}.$$

The corresponding variance-covariance matrix can be estimated as

$$V(\widehat{\beta}_s) = \sigma_s^2 \big[\sum_t \theta(t-s) Z_{it} Z'_{it}\big]^{-1},$$

where σ_s^2 is the error variance within a local neighborhood of X_s that is approximately stationary.

Finally, letting s move along time we obtain a local regression $f(X_{-i}) = Z_i \widehat{\beta}_s$ and the corresponding residual weighted sum of squares $RwSS_s = \sum_t \theta(t-s) \big[Z'_{it} \widehat{\beta}_s - x_{it}\big]^2$. These residuals could then be used to calculate a series of local coefficients of determination

$$R_s^2 = 1 - \frac{RwSS_s}{TwSS_s}, \quad s = 1 \ldots T, \tag{1}$$

where $TwSS_s = \sum_t \theta(t-s)\, x_{it}^2$ is the total weighted sum of squares at time s. Alternatively, they can also be obtained directly from the data correlation matrix as in Fernández-Macho (2012) (see Section 4).

4. Wavelet local multiple regression

Let $W_{jt} = (w_{1jt}, x_{2jt}, \ldots, w_{njt})$ be the scale λ_j wavelet coefficients obtained by applying the maximal overlap discrete wavelet transform (MODWT) (Gençay et al., 2002; Percival and Walden, 2000) to each time series $x_i \in X$ in Section 3. Fernández-Macho (2012) used a MODWT based global multiple regression context to define the wavelet multiple correlation (WMC) $\varphi_X(\lambda_j)$, and its companion wavelet multiple cross-correlation (WMCC), as single sets of multiscale correlations and cross-correlations calculated for the multivariate time series X. However, the evolution of nonstationary dynamics cannot be studied with these global measures and therefore they need to be extended to a local multiple regression framework where comovement dynamics can be properly analyzed.

Following Fernández-Macho (2012), at each wavelet scale λ_j we calculate a series of local multiple correlation coefficients as the square roots of the regression coefficients of determination (Eq. (1)) for that linear combination of variables w_{ijt}, i=1… n, where such coefficients of determination are maxima. In practice, none of these auxiliary regressions need to be run since, as it is well known, the coefficient of determination corresponding to the regression of a variable z_i on a set of regressors $\{z_k,\ k \neq i\}$, can be obtained as

$R_i^2 = 1 - 1/\rho^{ii}$, where ρ^{ii} is the i-th diagonal element of the inverse of the complete correlation matrix P. Therefore, $\varphi_{X,s}(\lambda_j)$ is obtained as

$$\varphi_{X,s}(\lambda_j) = \sqrt{1 - \frac{1}{\max \operatorname{diag} P_{j,s}^{-1}}}, \quad s = 1 \ldots T, \tag{2}$$

where $P_{j,s}$ is the $(n \times n)$ weighted correlation matrix of W_{jt} with weights $\theta(t-s)$, and the $\max \operatorname{diag}(\cdot)$ operator selects the largest element in the diagonal of the argument.

Since a regression coefficient of determination can also be obtained as the square of the correlation between observed values and fitted values, we have that $\varphi_{X,s}(\lambda_j)$ can also be expressed as

$$\begin{aligned}\varphi_{X,s}(\lambda_j) &= \operatorname{Corr}(\theta(t-s)^{1/2} w_{ijt}, \theta(t-s)^{1/2} \widehat{w}_{ijt}) \\ &= \frac{\operatorname{Cov}(\theta(t-s)^{1/2} w_{ijt}, \theta(t-s)^{1/2} \widehat{w}_{ijt})}{\sqrt{\operatorname{Var}(\theta(t-s)^{1/2} w_{ijt}) \operatorname{Var}(\theta(t-s)^{1/2} \widehat{w}_{ijt})}}, \quad s = 1 \ldots T,\end{aligned} \tag{3}$$

where w_{ij} is chosen so as to maximize $\varphi_{X,s}(\lambda_j)$ and $\widehat{w}_{ij}$ are the fitted values in the local regression of w_{ij} on the rest of wavelet coefficients at scale λ_j. Hence the adopted name of 'wavelet local multiple correlation' (WLMC) for this new statistic. Equation (3) will be useful later in determining the statistical properties of an estimator of $\varphi_{X,s}(\lambda_j)$.

Applying a MODWT of order J to each of the univariate time series in X we would obtain J length-T vectors of MODWT coefficients $\widetilde{W}_j = \{\widetilde{W}_{j0} \ldots \widetilde{W}_{j,T-1}\}$, for j=1… J. From Eq. (2) the WLMC of scale λ_j is a nonlinear function of all the $n(n-1)/2$ weighted correlations of W_{jt}. Alternatively, it can also be expressed in terms of all the weighted covariances and variances of W_{jt} as in Eq. (3). Therefore, a consistent estimator of the WLMC based on the MODWT is given by

$$\begin{aligned}\widetilde{\varphi}_{X,s}(\lambda_j) &= \sqrt{1 - \frac{1}{\max \operatorname{diag} \widetilde{P}_{j,s}^{-1}}} \\ &= \operatorname{Corr}(\theta(t-s)^{1/2} \widetilde{w}_{ijt}, \theta(t-s)^{1/2} \widehat{\widetilde{w}}_{ijt}) \\ &= \frac{\operatorname{Cov}(\theta(t-s)^{1/2} \widetilde{w}_{ijt}, \theta(t-s)^{1/2} \widehat{\widetilde{w}}_{ijt})}{\sqrt{\operatorname{Var}(\theta(t-s)^{1/2} \widetilde{w}_{ijt}) \operatorname{Var}(\theta(t-s)^{1/2} \widehat{\widetilde{w}}_{ijt})}}, \quad s = 1 \ldots T,.\end{aligned} \tag{4}$$

The weighted wavelet covariances and variances can be estimated as

$$\mathrm{Cov}(\widetilde{w}_{ijt}, \widehat{\widetilde{w}}_{ijt}) = \gamma_{j,s} = \sum_{t=L_j-1}^{T-1} \theta(t-s)\, \widetilde{w}_{ijt}\, \widehat{\widetilde{w}}_{ijt}, \quad s = 1 \ldots \widetilde{T}, \tag{5a}$$

$$\mathrm{Var}(\widetilde{w}_{ijt}) = \delta_{j,s}^2 = \sum_{t=L_j-1}^{T-1} \theta(t-s)\, \widetilde{w}_{ijt}^2, \quad s = 1 \ldots \widetilde{T}, \tag{5b}$$

$$\mathrm{Var}(\widehat{\widetilde{w}}_{ijt}) = \zeta_{j,s}^2 = \sum_{t=L_j-1}^{T-1} \theta(t-s)\, \widehat{\widetilde{w}}_{ijt}^2, \quad s = 1 \ldots \widetilde{T}, \tag{5c}$$

where $\widetilde{w}_{ij}$ is such that the local regression of $\widetilde{w}_{ij}$ on the set of regressors $\{\widetilde{w}_{kj}, k \neq i\}$ maximizes the corresponding coefficient of determination, $\widehat{\widetilde{w}}_{ij}$ denotes the fitted values and $L_j = (2^j - 1)(L-1) + 1$ is the number of wavelet coefficients affected by the boundary associated with a wavelet filter of length L at scale λ_j.

The large-sample distribution of the sample WLMC $\widetilde{\varphi}_{X,s}(\lambda_j)$ can be established along similar lines as for the standard pairwise wavelet correlation in Gençay et al. (2002). In our present local multivariate case, we note from Eq. (4) that $\widetilde{\varphi}_{X,s}(\lambda_j)$ is a nonlinear function of all the sample weighted wavelet variances and covariances which, in turn, are just sample moments of vectors of MODWT coefficients. Therefore, the estimator can be written as a function of the three local wavelet moments in Eq. (5):

$$\widetilde{\varphi}_{X,s}(\lambda_j) = f(\gamma_{j,s}, \delta_{j,s}, \zeta_{j,s}) = \frac{\gamma_{j,s}}{\delta_{j,s}\, \zeta_{j,s}}.$$

We may now apply the continuous mapping theorem to establish that

$$\sqrt{\widetilde{T}_j}\left[\widetilde{\varphi}_{X,s}(\lambda_j) - \varphi_{X,s}(\lambda_j)\right] \sim \mathcal{N}(0, V_{j,s}),$$

where $\widetilde{T}_j = T - L_j + 1$ is the number of coefficients unaffected by the boundary conditions and

$$V_{j,s} = df_{j,s}'\, S_{j,s}(0)\, df_{j,s},$$

with $df_{j,s}$ as the gradient vector of $f(\gamma_{j,s}, \delta_{j,s}, \zeta_{j,s})$ and

$$S_{j,s}(0) = \begin{pmatrix} S_{\gamma^2,j,s}(0) & S_{\delta\gamma,j,s}(0) & S_{\zeta\gamma,j,s}(0) \\ S_{\delta\gamma,j,s}(0) & S_{\delta^2,j,s}(0) & S_{\zeta\delta,j,s}(0) \\ S_{\zeta\gamma,j,s}(0) & S_{\zeta\delta,j,s}(0) & S_{\zeta^2,j,s}(0) \end{pmatrix}$$

where *e.g.* $S_{\delta\gamma,j,s}(0)$ is the spectral density function of the product of scale λ_j local wavelet moments $\delta_{j,s}\gamma_{j,s}$ evaluated at the zero frequency, *etc.*(*cf.* Whitcher, 1998).

In practice, obtaining the spectral density functions involved in the computation of a consistent estimate of $V_{j,s}$ can be very tiresome. As in Gençay et al. (2002) or Fernández-Macho (2012), a more feasible alternative can be obtained by using Fisher (1915)'s transformation. In the present case, applying Fisher's transformation to the sample wavelet multiple correlation coefficient $\widetilde{\varphi}_X(\lambda_j)$ we obtain:

Theorem 1. *Let $X = \{X_1 \dots X_T\}$ be a realization of a multivariate Gaussian stochastic process $X_t = (x_{1t}, x_{2t}, \dots, x_{nt})$ and let $\widetilde{W}_j = \{\widetilde{W}_{j0} \dots \widetilde{W}_{j,T-1}\} = \{(\widetilde{w}_{1j0} \dots \widetilde{w}_{nj0}), \dots, (\widetilde{w}_{1j,T/2^j-1} \dots \widetilde{w}_{nj,T/2^j-1})\}$, j=1... J, be vectors of wavelet coefficients obtained by applying a MODWT of order J to each of the univariate time series $\{x_{i1} \dots x_{iT}\}$ for i=1... n. Let $\widetilde{\varphi}_{X,s}(\lambda_j)$ be the sample wavelet local multiple correlation (WLMC) obtained from Eq.* (2). *Then,*

$$\widetilde{z}_{j,s} \overset{a}{\sim} \mathcal{FN}(z_{j,s}, (T/2^j - 3)^{-1}).$$

where $z_{j,s} = \operatorname{arctanh}(\varphi_{X,s}(\lambda_j))$ and $\mathcal{FN}$ stands for the folded normal distribution.[3]

The demonstration is straightforward along similar lines as in Fernández-Macho (2012). X being Gaussian implies that, at each scale λ_j, the sample wavelet coefficients in $\widetilde{W}_j$ are also Gaussian and, in turn, this means that $\widetilde{\widetilde{w}}_{ij}$, which is a linear combination of $\widetilde{w}_{1j}, \dots, \widetilde{w}_{nj}$, must also be Gaussian. Therefore, we have from Eq. (3) that $\widetilde{\varphi}_{X,s}(\lambda_j)$ is a correlation coefficient between weighted observations from two Gaussian variates, of which $T/2^j$ are (asymptotically) serially uncorrelated.[4] Applying Fisher's result the theorem follows. □

Therefore,

$$CI_{(1-\alpha)}(\varphi_{X,s}(\lambda_j)) = \tanh\left[\widetilde{z}_{j,s} \pm \phi^{-1}_{1-\alpha/2} / \sqrt{T/2^j - 3}\right],$$

where ϕ_p^{-1} is the $100p\%$ point of the standard normal distribution, can be used in practice to construct a confidence interval for the wavelet multiple correlation coefficient, as well as for testing hypothesis about the value of the wavelet correlation amongst a multivariate set of observed variables X.

5. Weight functions and their spectral properties

In principle, any weight function satisfying the properties suggested by Cleveland (1979) could be used and in fact many choices have been proposed in the literature (see *e.g.* Priestley, 1981). Table 1 shows a selection of the six most widely used functions for averaging and smoothing that are graphically depicted in Fig. 1. It can be shown that for all of them $\int_{-\infty}^{\infty} \theta(x)dx = 1$, so that they can be described as moving averages.

However, the spectral properties of these moving averages differ much as shown by the corresponding Fourier transforms in Figs. 2 and 3. For example, the Fourier transform of the variances needed to calculate the wavelet local correlations $\varphi_X(\lambda_j)$ in Eq. (3) can

[3] That is, the probability distribution of abs(ϵ) such that ϵ is normally distributed with the given mean and variance.

[4] Note that this is the number of wavelet coefficients from a DWT that can be shown to decorrelate a wide range of stochastic processes (Craigmile and Percival, 2005).

Table 1: Weight functions

	solid	long dash	short dash
blue lines:	Uniform window: $u(x) = 1/(2\,M)$	Cleveland's tricube window: $c(x) = 70\big(1 - \lvert\frac{x}{M}\rvert^3\big)^3/(81\,M)$	Epanechnikov's parabolic window: $e(x) = 3\big(1 - \lvert\frac{x}{M}\rvert^2\big)/(4\,M)$
red lines:	Bartlett's triangular window: $b(x) = \big(1 - \lvert\frac{x}{M}\rvert\big)/M$	Wendland's truncated power window: $w(x) = 3\big(1 - \lvert\frac{x}{M}\rvert\big)^4\big(4\lvert\frac{x}{M}\rvert + 1\big)/(2\,M)$	Gaussian window: $g(x) = e^{-(x/M)^2}/(\sqrt{\pi}\,M)$

All windows $\theta(x)$ with compact support $|x| \leq M$ and 0 value otherwise with $M > 0$, except the Gaussian window that takes values for $x \in (-\infty, \infty)$. The line color and format refer to their graphical representation in Fig. 1.

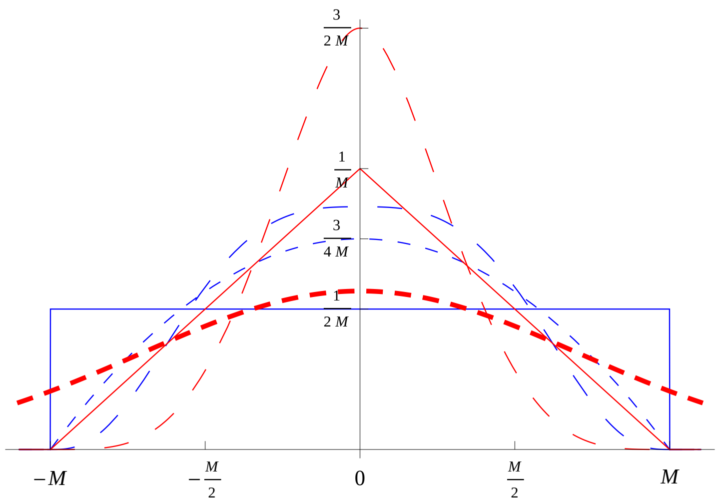

Figure 1: Some weight functions for local least-squares regression (see Table 1). Blue lines: $u(x)$ solid, $c(x)$ long-dash, $e(x)$ short-dash; Red lines: $b(x)$ solid, $w(x)$ long-dash, $g(x)$ thick short-dash.

be obtained from Eq. (5b) as

$$\tau(\omega) = \frac{1}{2\pi} \sum_{s=1}^{\widetilde{T}} e^{-i\omega s} \delta_{j,s}^2 = \frac{1}{\sqrt{2\pi}} \sum_{s=1}^{\widetilde{T}} e^{-i\omega s} \frac{1}{\sqrt{2\pi}} \sum_{t=L_j-1}^{T-1} \theta(t-s)\, \widetilde{w}_{ijt}^2$$
$$= \frac{1}{\sqrt{2\pi}} \sum_{t=L_j-1}^{T-1} e^{-i\omega t} \widetilde{w}_{ijt}^2 \Big[\frac{1}{\sqrt{2\pi}} \sum_{s=1-t}^{T-t} e^{-i\omega s} \theta(s) \Big] = \xi(\omega)\phi(\omega), \quad \omega \in (-\pi, \pi),$$

where $\phi(\omega)$, the Fourier transform of the weight function $\theta(x)$, must be non-negative to ensure that $\tau(\omega) \geq 0$. This is certainly a desirable property for the spectral decomposition of a variance (*cf.* Priestley, 1981, p.438) and similarly for $\zeta_{j,s}^2$ and $\gamma_{j,s}$ in Eq. (5).

However, common weight functions such as the uniform window $u(x)$, Cleveland (1979)'s tricube window $c(x)$ and Epanechnikov (1969)'s parabolic window $e(x)$ are not recommended because of the presence of negative values in their corresponding spectral windows. This means that, within a particular timescale, some frequencies may be negatively weighted, which is not desirable. Even more so, those frequencies that are negatively weighted are not invariant to the choice of bandwidth $2\pi/M$ so that, in the present case, the sign of correlation at frequencies corresponding to a particular timescale may change depending of the choice of bandwidth. This, as can be seen in Fig. 2, is particularly true for the uniform window which is a common choice in the rolling-window wavelet correlation literature so far.

On the other hand, some other weight functions such as the Gaussian window, Bartlett (1950)'s triangular window and Wendland (1995)'s truncated power window are favored in the signal extraction and smoothing literature because their spectral windows are non-negative everywhere which in the present case ensures that the sign of correlation at frequencies corresponding to a particular timescale will be invariant to the choice of bandwidth.

If computational power is an issue Bartlett and Wendland windows are probably good choices that are not difficult to implement. In what follows, we will use the Gaussian window because of some desirable properties. In particular, it is the closest to uniform weights in the time domain within a given bandwidth, its Fourier transform is also a Gaussian function, it has near-compact support in the frequency domain, and its spectral window is positive everywhere, therefore frequency/scale weights are always positive and correlation sign is invariant to the choice of bandwidth.

6. Example with breaks in the correlation structure

Figure 4 shows the WLMC of a bivariate simulated time series $X_t = (x_t, y_t)$ of length $T = 512$ with varying correlation both along time and across timescales/frequencies. The

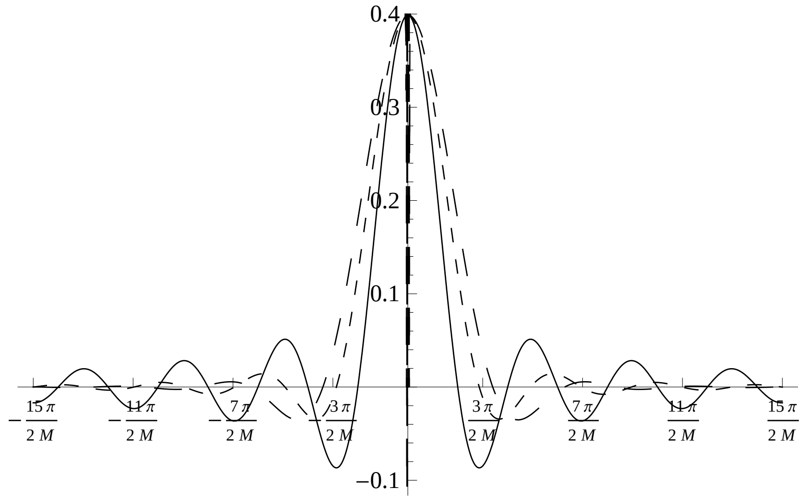

Figure 2: Spectral windows corresponding to the blue weight functions in Fig. 1: $u(x)$ solid, $c(x)$ long-dash, $e(x)$ short-dash. Note that they take negative values at some frequencies.

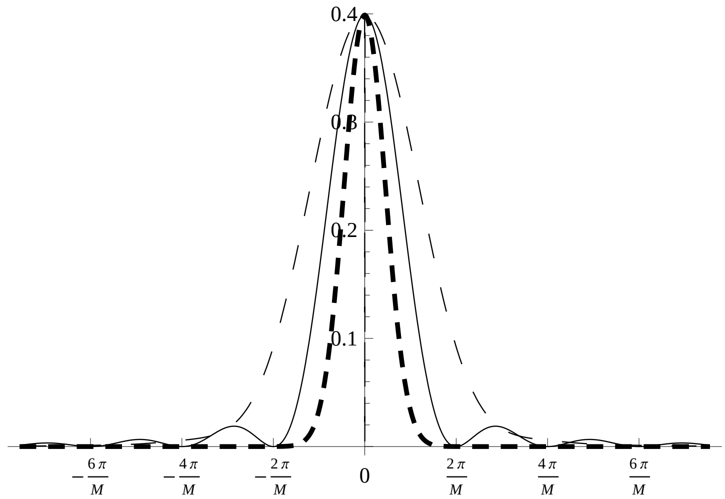

Figure 3: Spectral windows corresponding to the red weight functions in Fig. 1: $b(x)$ solid, $w(x)$ long-dash, $g(x)$ thick short-dash. Note that they are non-negative everywhere.

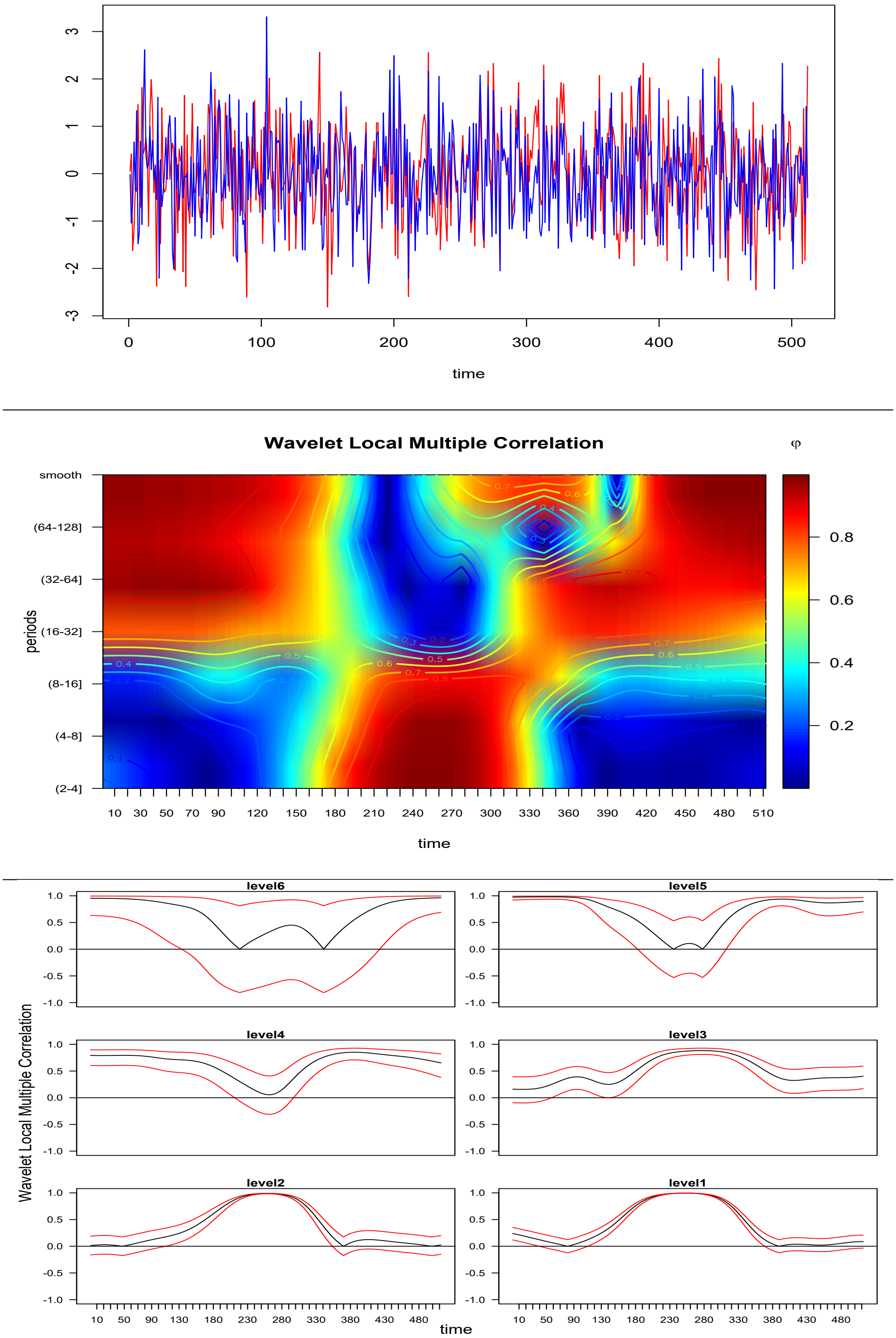

Figure 4: Wavelet local multiple correlation of two simulated time series (top) with varying correlation (middle). The red lines (bottom) correspond to the upper and lower bounds of the 95% confidence interval at different timescales.

two series were generated as follows,

$$
\begin{aligned}
x_t &= \eta_{1t}, \\
y_t(\omega_1) &= \begin{cases} \eta_{2t}, \ T/3 < t < 2T/3, \\ .9\,\eta_{1t}, \ \text{otherwise}, \end{cases} \quad , \quad \omega_1 \in (0, \pi/8), \\
y_t(\omega_2) &= \begin{cases} .9\,\eta_{1t}, \ T/3 < t < 2T/3, \\ \eta_{2t}, \ \text{otherwise}, \end{cases} \quad , \quad \omega_2 \in [\pi/8, \pi),
\end{aligned}
$$

where η_{1t}, η_{2t} are two uncorrelated random variates generated by the normal distribution with zero mean and variance 0.1. That is, x_t and y_t are highly correlated at low frequencies (long timescales) but uncorrelated at high frequencies (short timescales). However, during a period of time spanning the second third of the sample ($T/3 < t < 2T/3$) that behavior is reversed so that data become highly correlated at short timescales but uncorrelated at low frequencies. The heatmap in the middle of Fig. 4 shows how the WLMC clearly picks up these changes while the graphs at the bottom show the 95% confidence intervals for the correlations at each individual scale.

7. Eurozone stock markets

In this section we illustrate the usage of the advocated wavelet local multiple correlation (WLMC) with data from the eleven main Eurozone stock markets as follows (ordered by nominal GDP of the country where they operate): DAX (Germany), CAC40 (France), FTSE/MIB (Italy), IBEX35 (Spain), AEX (Netherlands), BEL20 (Belgium), ATX (Austria), ISEQ (Ireland), OMXH25 (Finland), PSI20 (Portugal) and FTSE/ASE (Greece). The data were collected daily (closing prices) from January 3, 2000 (Monday) to May 31, 2017 (Wednesday)[5] and the analysis was conducted using daily stock market returns, *i.e.*, $R_{it} = \log(S_{it}/S_{i,t-1}) = \Delta \log S_{it}$, where S_{it}, i=1...11, t=2...4543, are the corresponding stock market index values. Therefore, the total number of observations used is 49962 trading days, thus containing a large amount of information that may not be easy to convey using standard procedures.

In order to calculate the proposed WLMC we decomposed the daily stock market returns applying the MODWT with a Daubechies wavelet filter of length L=4 (Daubechies, 1992).[6] The maximum decomposition level J is given by $\lfloor \log_2(T) \rfloor$ (Percival and Walden, 2000), which, in the present case, means a maximum level of 12. Since the number of feasible wavelet coefficients gets critically small for high levels, we chose to carry out the wavelet analysis with J=9 so that ten vectors (nine of wavelet coefficients and one of

[5] As published by Yahoo! https://finance.yahoo.com, Investing https://www.investing.com/indices/ and Financial Times https://markets.ft.com/data/indices.

[6] Results with the Daubechies LA(8) filter were practically the same except for the two longest timescales (one year and longer) that were indistinguishable from 1 up to three decimal points for the whole length of the sample period.

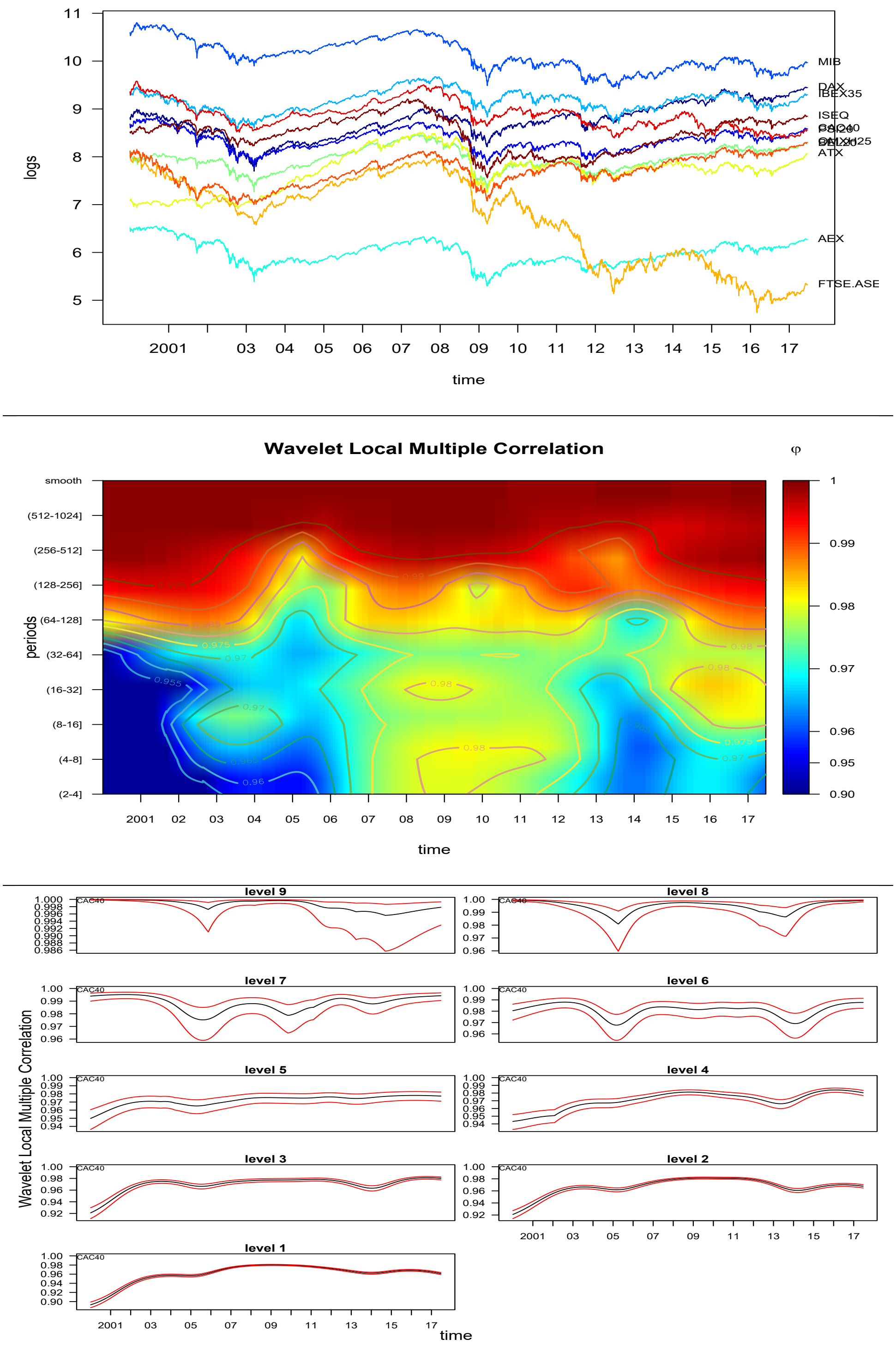

Figure 5: EUro11 Stocks and WMLC of their daily returns.

scaling coefficients) were produced for each daily returns series, *i.e.* $\widetilde{w}_{i1}, \ldots, \widetilde{w}_{i9}$ and $\widetilde{v}_{i9}$, i=1... 11, respectively.

We may note that, since a MODWT based on Daubechies wavelets approximates an ideal band-pass filter with bandpass given by the frequency interval $[2^{-j}\pi, 2^{1-j}\pi)$ for j=1... J, after inverting that frequency range the corresponding periods are within $(2^j, 2^{j+1}]$ time units intervals (Whitcher et al., 2000). This means that, with 5 daily data per week, the scales λ_j, j=1... 9, of the wavelet coefficients are associated to periods of, respectively, 2–4 days (which includes most intraweek scales), 4–8 days (including the weekly scale), 8–16 days (fortnightly scale), 16–32 days (monthly scale), 32–64 days (quarterly scale), 64–128 days (quarterly to biannual scale), 128–256 days (biannual scale), 256–512 days (annual scale) and 512–1028 days (two to four year scale).

Figure 5 shows the wavelet local multiple correlations obtained as a measure of comovement dynamics among Eurozone stock markets. As can be seen, the correlation structure of these markets during this century is far from homogeneous both along time and across timescales/frequencies. One of the most striking features that the WLMC brings to light is the sharp divide in the correlation structure across timescales at about the quarterly scale.

Above the quarterly scale, the long-term correlation structure appears to be quite stable in time with multiple correlations that are all very high starting at around 0.98 and reaching values near 1 at the longest timescales (year long and above). This can be interpreted as near integration among Euro stock markets at quarterly horizons and above in the sense that the returns obtained in any of them can be determined by the overall performance in the other markets (Syllignakis, 2006; Li and Daly, 2014). Also, for time horizons of one year and longer the existence of an exact linear relationship between Eurozone stock markets cannot be ruled out. This can be taken as evidence of perfect integration even in spite of several financial and debt crises occurring during this period.

On the other hand, the WLMC in Fig. 5 shows that for intraweek and intramonth periods the correlation structure is clearly evolving along time. This is specially so during the financial crises of 2007–2012. Prior to this period, short-term multiple correlations were around 0.90 but experienced a sharp increase above 0.98 during 2007 which can be taken as evidence of financial 'contagion' (Forbes and Rigobon, 2002; Duffie et al., 2009; Ranta, 2013). Such phenomenon appears to have mitigated after 2012 where short-term multiple correlations have decreased.

8. Conclusions

This paper presents a new statistical tool, the wavelet local multiple correlation (WLMC), that may be useful in the analysis of comovements within a set of time series. The WLMC consists in one single set of multiscale correlations along time, each of them calculated as the square root of the regression coefficient of determination in that linear combination of locally weighted wavelet coefficients for which such coefficient of determination is a maximum. In contrast, the alternative of using standard bivariate wavelet correlations with rolling windows needs to calculate, plot and compare a large number of wavelet correlation maps and it may run out of data points at longer timescales. Also, it has been shown how the spectral properties of the weight function or rolling window need to be taken into account.

Figures 4 and 5 offer some graphical examples of this tool as obtained in a simulated data set with breaks in the correlation structure and in the time-localized wavelet analysis of a set of eleven Eurozone stock market returns during a recent period of 4542 trading days. The WLMC analysis reveals the existence of a stable and practically exact linear relationship between these stock markets for periods of time of one year and longer, which can be interpreted as perfect integration. On the other hand, the WLMC shows that for intraweek and intramonth periods the correlation structure is clearly evolving along time, experiencing a sharp increase during financial crises which some authors interpret as evidence of financial 'contagion'. We may finally point out that all these results would be quite hard to establish using the standard wavelet analysis that relies on the visualization of all the 55 wavelet correlation maps between pairs of variables and they serve to illustrate the potential of this new tool in the multiscale analysis of multivariate data along time.

Supplemental material

A new version of the wavemulcor R computer package facilitates the computation of wavelet local multiple correlations. It can be obtained from The Comprehensive R Archive Network (CRAN) at `http://cran.r-project.org/web/packages/wavemulcor/index.html` or directly from the author upon request.

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