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# Robust Resampling Methods for Time Series\*

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## Abstract

We study the robustness of block resampling procedures for time series. We first derive a set of formulas to characterize their quantile breakdown point. For the moving block bootstrap and the subsampling, we find a very low quantile breakdown point. A similar robustness problem arises in relation to data-driven methods for selecting the block size in applications. This renders inference based on standard resampling methods useless already in simple estimation and testing settings. To solve this problem, we introduce a robust fast resampling scheme that is applicable to a wide class of time series settings. Monte Carlo simulations and sensitivity analysis for the simple AR(1) model confirm the dramatic fragility of classical resampling procedures in presence of contaminations by outliers. They also show the better accuracy and efficiency of the robust resampling approach under different types of data constellations. A real data application to testing for stock return predictability shows that our robust approach can detect predictability structures more consistently than classical methods.

**Keywords:** Subsampling, bootstrap, breakdown point, robustness, time series.

**JEL:** C12, C13, C15.

**MSC 2000:** Primary 62F40; Secondary 62F35.

# 1 Introduction

Resampling methods, including the bootstrap (see, e.g., Hall, 1992, Efron and Tibshirani, 1993, and Hall and Horowitz, 1996) and the subsampling (see, e.g., Politis and Romano, 1992, 1994a, Politis, Romano and Wolf, 1999), are useful tools in modern statistics and econometrics. The simpler consistency conditions and the wider applicability in some cases (see, e.g., Andrews, 2000, and Bickel, Gotze and van Zwet, 1997) make the subsampling a useful and valid alternative to the bootstrap in a number of statistical models. Bootstrap and subsampling procedures for time series typically rely on different block resampling schemes, in which selected sub-blocks of the data, having size strictly less than the sample size, are randomly resampled. This feature is necessary in order to derive consistent resampling schemes under different assumptions on the asymptotically vanishing time series dependence between observations. See, among others, Hall (1985), Carlstein (1986), Künsch (1989), Politis, Romano and Wolf (1999), Bühlmann (2002), and Lahiri (2003).

The low robustness of classical bootstrap and subsampling methods is a known feature in the iid setting; see, e.g., Singh (1998), Salibian-Barrera and Zamar (2002), Salibian-Barrera, Van Aelst and Willems (2006, 2007), and Camponovo, Scaillet and Trojani (2009). These papers study global robustness features and highlight a typically very low breakdown point of classical bootstrap and subsampling quantiles. Moreover, they show that this robustness problem cannot be exhausted simply by applying classical bootstrap and subsampling to high breakdown point statistics. Essentially, the breakdown point quantifies the smallest fraction of outliers in the data which makes a statistic meaningless. Therefore, standard iid resampling methods produce es-

estimated quantiles that are heavily dependent on a few possible outliers in the original data. Intuitively, this lack of robustness is related to the (typically high) probability of resampling a large number of outliers in a random sample using an iid bootstrap or subsampling scheme. To overcome this problem, robust bootstrap and subsampling approaches with desirable quantile breakdown point properties have been developed in the iid context by Salibian-Barrera and Zamar (2002), Salibian-Barrera, Van Aelst and Willems (2006,2007), and Camponovo, Scaillet and Trojani (2009), among others.

In this paper, we study the robustness of block resampling methods for time series and develop fast robust resampling approaches that are applicable to a variety of time series models. We first characterize the breakdown properties of block resampling procedures for time series by deriving lower and upper bounds for their quantile breakdown point; these results cover both overlapping and nonoverlapping bootstrap and subsampling procedures. Concrete computations show that block resampling methods for time series suffer of an even larger robustness problem than in the iid context. This problem cannot be mitigated simply by applying standard block resampling methods to a more robust statistic, indicating the high need for a more robust resampling scheme applicable in the time series context.

We develop our robust resampling approach for time series following the fast resampling idea put forward, among others, in Shao and Tu (1995), Davidson and McKinnon (1999), Hu and Kalbfleisch (2000), Andrews (2002), Salibian-Barrera and Zamar (2002), Goncalves and White (2004), Hong and Scaillet (2006), Salibian-Barrera, Van Aelst and Willems (2006, 2007), and Camponovo, Scaillet and Trojani (2009). Our resampling method is applicable to a wide class of resampling procedures, including both the block bootstrap

and the subsampling; it provides robust estimation and inference results under weak conditions. Moreover, it inherits the low computational cost of fast resampling approaches. This makes it applicable to nonlinear models when classical resampling methods might become computationally too expensive, or in combination with computationally intensive data-driven procedures for the selection of the optimal block size; see, for instance, Sakata and White (1998), Ronchetti and Trojani (2001), Mancini, Ronchetti and Trojani (2005), Ortelli and Trojani (2005), Muler and Yohai (2008), and La Vecchia and Trojani (2010) for recent examples of robust estimators for nonlinear time series models. By means of explicit breakdown point computations, we also find that the better breakdown properties of our fast robust resampling scheme are inherited by data-driven choices of the block size based on either the minimum confidence index volatility (MCIV) and the calibration method (CM), proposed in Romano and Wolf (2001) for the subsampling, or the data-driven method in Hall, Horowitz and Jing (1995) (HHJ) for the moving block bootstrap.

We investigate by Monte Carlo simulations the performance of our robust resampling approach in the benchmark context of the estimation of the autoregressive parameter in an AR(1) model both in a strictly stationary and near-to-unit root setting. Overall, our Monte Carlo experiments highlight a dramatic fragility of classical resampling methods in presence of contaminations by outliers, and a more reliable and efficient inference produced by our robust resampling method under different types of data constellations. Finally, in an application to real data, we find that our robust resampling approach detects predictability structures in stock returns more consistently than standard methods.

The paper is organized as follows. Section 2 outlines the main setting and

introduces the quantile breakdown point formulas of different block resampling procedures. In Section 3, we develop our robust approach and derive the relevant expression for the associated quantile breakdown point formula. We show that, under weak conditions, the resulting quantile breakdown point is maximal. In Section 4, we study the robustness properties of data-driven block size selection procedures based on the MCIV, the CM, and the HHJ method. Monte Carlo experiments, sensitivity analysis, and the empirical application to stock return predictability are presented in Section 5. Section 6 concludes.

## 2 Resampling Distribution Breakdown Point Quantile

We start our analysis by characterizing the robustness of resampling procedures for time series and by deriving formulas for their quantile breakdown point.

### 2.1 Definition

Let  $X_{(n)} = (X_1, \dots, X_n)$  be a sample from a real valued stationary process  $\mathcal{X} = \{X_t, t \in \mathbb{Z}\}$  defined on the probability space  $(\Omega, \mathcal{F}, P)$ , and consider a real valued statistic  $T_n := T(X_{(n)})$ .

In the time series setting, block bootstrap procedures split the original sample in overlapping or nonoverlapping blocks of size  $m < n$ . Then, new random samples of size  $n$  are constructed assuming an approximate independence between blocks. Finally, the statistic  $T$  is applied to the so generated random samples; see, e.g., Hall (1985), Carlstein (1986), Künsch (1989), and



Andrews (2004). The more recent subsampling method (see, e.g., Politis, Romano and Wolf, 1999), instead, directly applies statistic  $T$  to overlapping or nonoverlapping blocks of size  $m$  strictly less than  $n$ .

Let  $X_{(k)}^* = (X_1^*, \dots, X_k^*)$  denote for brevity a bootstrap ( $k = n$ ) or a subsampling ( $k = m < n$ ) random sample and  $T_{n,k}^* := T(X_{(k)}^*)$  be the bootstrap or subsampling statistic, respectively. Then, for  $t \in (0, 1)$ , the quantile  $Q_t^*$  of  $T_{n,k}^*$  is defined by

$$Q_t^* = \inf\{x | P^*(T_{n,k}^* \leq x) \geq t\}, \quad (1)$$

where  $P^*$  is the corresponding bootstrap or subsampling distribution and, by definition,  $\inf(\emptyset) = \infty$ .

We characterize the robustness of quantile (1) via its breakdown point, i.e., the smallest fraction of outliers in the original sample such that  $Q_t^*$  degenerates, making inference based on (1) meaningless. Different than in the iid case, in time series we can consider different possible models of contamination by outliers, like for instance additive outliers, replacement outliers, and innovation outliers; see, e.g., Martin and Yohai (1986). Because of this additional complexity, we first introduce a notation that can better capture the effect of such contaminations, following Genton and Lucas (2003). Denote by  $\mathcal{Z}_p^\zeta$  the set of all  $n$ -components outlier samples, where  $p$  is the number of outliers and index  $\zeta \in \bar{R}$  indicates their size. When  $p > 1$ , we do not necessarily assume outliers  $\zeta_1, \dots, \zeta_p$  to be all equal to  $\zeta$ , but we rather assume existence of constants  $c_1, \dots, c_p$ , such that  $\zeta_i = c_i \zeta$ .

Let  $0 \leq b \leq 0.5$  be the upper breakdown point of statistic  $T_n$ , i.e.,  $nb$  is the smallest number of outliers such that  $T(X_{(n)} + Z_{nb}^\zeta) = +\infty$  for some  $Z_{nb}^\zeta \in \mathcal{Z}_{nb}^\zeta$ . The breakdown point  $b$  is an intrinsic characteristic of a statistic.

It is explicitly known in some cases and it can be gauged most of the time, for instance by means of simulations and sensitivity analysis. In this section, we focus for brevity on one-dimensional real valued statistics. As discussed for instance by Singh (1998) in the iid context, our quantile breakdown point results for time series can be naturally extended to consider multivariate and scale statistics. Formally, the quantile breakdown point of  $Q_t^*$  is defined as follows:

**Definition 1** *The upper breakdown point of the  $t$ -quantile  $Q_t^*$  is given by*

$$b_t := \frac{1}{n} \cdot \left[ \inf_{\{1 \leq p \leq \lceil n/2 \rceil\}} \{p \mid \text{there exists } Z_p^\zeta \in \mathcal{Z}_p^\zeta \text{ such that } Q_t^*(X_{(n)} + Z_p^\zeta) = +\infty\} \right], \quad (2)$$

where  $\lceil x \rceil = \inf\{n \in \mathbb{N} \mid x \leq n\}$ .

## 2.2 Quantile Breakdown Point

We derive lower and upper bounds for the quantile breakdown point of the overlapping subsampling and both nonoverlapping and overlapping moving block bootstrap procedures. Similar results can be obtained for the nonoverlapping subsampling. Since that case is of little practical interest, because unless the sample size is very large the number of blocks is too small to make reliable inference, we do not report results for such a case. For brevity, we denote by  $b_t^K$ ,  $K = OS, NB, OB$ , the upper  $t$ -quantile breakdown point of the overlapping subsampling and the nonoverlapping and overlapping moving block bootstrap, respectively. Results for the overlapping moving block bootstrap can be modified to cover asymptotically equivalent variations such as the stationary bootstrap of Politis and Romano (1994b).

### 2.2.1 Subsampling

For simplicity, let  $n/m = r \in \mathbb{N}$ . The overlapping subsampling splits the original sample  $X_{(n)} = (X_1, \dots, X_n)$  into  $n - m + 1$  overlapping blocks of size  $m$ ,  $(X_i, \dots, X_{i+m-1})$ ,  $i = 1, \dots, n - m + 1$ . Finally, it applies statistic  $T$  to these blocks.

**Theorem 2** *Let  $b$  be the breakdown point of  $T_n$  and  $t \in (0, 1)$ . The quantile breakdown point  $b_t^{OS}$  of overlapping subsampling procedures satisfies the following property:*

$$\frac{\lceil mb \rceil}{n} \leq b_t^{OS} \leq \inf_{\{p \in \mathbb{N}, p \leq r-1\}} \left\{ p \cdot \frac{\lceil mb \rceil}{n} \mid p > \frac{(1-t)(n-m+1) + \lceil mb \rceil - 1}{m} \right\}. \quad (3)$$

The term  $\frac{(1-t)(n-m+1)}{m}$  represents the number of degenerated statistics necessary in order to cause the breakdown of  $Q_t^*$ , while  $\frac{\lceil mb \rceil}{n}$  is the fraction of outliers which is sufficient to cause the breakdown of statistic  $T$  in a block of size  $m$ . In time series, the number of possible subsampling blocks of size  $m$  is typically lower than the number of iid subsamples of size  $m$ . Therefore, the breakdown of a statistic in one random block tends to have a larger impact on the subsampling quantile than in the iid case. Intuitively, this feature implies a lower breakdown point of subsampling quantiles in time series than in iid settings. Table 1 confirms this basic intuition. Using Theorem 2 we compute lower and upper bounds for the breakdown point of the overlapping subsampling quantile for a sample size  $n = 120$ , for  $b = 0.5$ , and for block sizes  $m = 5, 10, 15$ . We see that even for a maximal breakdown point statistic ( $b = 0.5$ ), the overlapping subsampling implies a very low quantile breakdown point, which is increasing in the block size, but very far from the maximal value  $b = 0.5$ . Moreover, this

breakdown point is clearly lower than in the iid case; see Camponovo, Scaillet and Trojani (2009). For instance, for  $m = 10$ , the 0.95-quantile breakdown point of the overlapping subsampling is lower than 0.05, which is less than a quarter of the breakdown point of 0.23 for the same block size in the iid setting.

### 2.2.2 Moving Block Bootstrap

Let  $X_{(m),i}^N = (X_{(i-1) \cdot m + 1}, \dots, X_{i \cdot m})$ ,  $i = 1, \dots, r$ , be the  $r$  nonoverlapping blocks of size  $m$ . The nonoverlapping moving block bootstrap selects randomly with replacement  $r$  nonoverlapping blocks  $X_{(m),i}^{N*}$ ,  $i = 1, \dots, r$ . Then, it applies statistic  $T$  to the  $n$ -sample  $X_{(n)}^{N*} = (X_{(m),1}^{N*}, \dots, X_{(m),r}^{N*})$ . Similarly, let  $X_{(m),i}^O = (X_i, \dots, X_{i+m-1})$ ,  $i = 1, \dots, n - m + 1$ , be the  $n - m + 1$  overlapping blocks. The overlapping moving block bootstrap selects randomly with replacement  $r$  overlapping blocks  $X_{(m),i}^{O*}$ ,  $i = 1, \dots, r$ . Then, it applies statistic  $T$  to the  $n$ -sample  $X_{(n)}^{O*} = (X_{(m),1}^{O*}, \dots, X_{(m),r}^{O*})$ .

**Theorem 3** *Let  $b$  be the breakdown point of  $T_n$ ,  $t \in (0, 1)$ , and  $p_1, p_2 \in \mathbb{N}$ , with  $p_1 \leq m, p_2 \leq r - 1$ . The quantile breakdown points  $b_t^{NB}$  and  $b_t^{OB}$  of the nonoverlapping and overlapping moving block bootstrap, respectively, satisfy the following properties:*

$$(i) \quad \frac{\lfloor mb \rfloor}{n} \leq b_t^{NB} \leq \frac{1}{n} \cdot \left[ \inf_{\{p_1, p_2\}} \left\{ p = p_1 \cdot p_2 \left| P \left( \text{BIN} \left( r, \frac{p_2}{r} \right) > \frac{nb}{p_1} \right) > 1 - t \right\} \right],$$

$$(ii) \quad \frac{\lfloor mb \rfloor}{n} \leq b_t^{OB} \leq \frac{1}{n} \cdot \left[ \inf_{\{p_1, p_2\}} \left\{ p = p_1 \cdot p_2 \left| P \left( \text{BIN} \left( r, \frac{mp_2 - p_1 + 1}{n - m + 1} \right) > \frac{nb}{p_1} \right) > 1 - t \right\} \right].$$

Similar to the findings for the subsampling, the right part of (i) and (ii) are similar for large  $n \gg m$ . Indeed, (ii) implies  $\frac{mp_2 - p_1 + 1}{n - m + 1} \approx \frac{mp_2}{n} = \frac{p_2}{r}$ , which is the right part of (i). Further the breakdown point formula for the iid bootstrap

in Singh (1998) emerges as a special case of the formulas in Theorem 3, for  $m = 1$ . This is intuitive: A nonoverlapping moving block bootstrap with block size  $m$  is essentially an iid bootstrap based on a sample of size  $r$ , in which each block of size  $m$  corresponds to a single random realization in the iid bootstrap. As for the subsampling, the reduction in the number of possible blocks when  $m \neq 1$  increases the potential impact of a contamination and it implies a lower quantile breakdown point. In Table 1, we compute lower and upper bounds for the breakdown point of the nonoverlapping and overlapping moving block bootstrap quantile for  $n = 120$ ,  $b = 0.5$ , and block sizes  $m = 5, 10, 15$ . Breakdown points are decreasing in the block size. Again, they are far from the maximal value  $b = 0.5$ . For instance, for  $m = 15$ , the 0.99-quantile breakdown point is less than 0.2126, which is approximatively half the breakdown point of 0.392 in the iid setting.

### 3 Robust Resampling Procedures

The results in the last section show that, even using statistics with maximal breakdown point, classical block resampling procedures imply a low quantile breakdown point. To overcome this problem, it is necessary to introduce a different and more robust resampling approach. We develop such robust resampling methods for M-estimators, starting from the fast resampling approach studied, among others, in Shao and Tu (1995), Davidson and McKinnon (1999), Hu and Kalbfleisch (2000), Andrews (2002), Salibian-Barrera and Zamar (2002), Goncalves and White (2004), Hong and Scaillet (2006), Salibian-Barrera, Van Aelst and Willems (2006,2007), and Camponovo, Scaillet and Trojani (2009).

### 3.1 Definition

Given the original sample  $X_{(n)} = (X_1, \dots, X_n)$ , we consider the class of robust M-estimators  $\hat{\theta}_n$  for parameter  $\theta \in \mathbb{R}^d$ , defined as the solution of the equations:

$$\psi_n(X_{(n)}, \hat{\theta}_n) := \frac{1}{n - q + 1} \sum_{i=q}^n g(X_{i-q+1}, \dots, X_i; \hat{\theta}_n) = 0, \quad (4)$$

where  $\psi_n(X_{(n)}, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$  depends on parameter  $\theta$  and a bounded estimating function  $g$ . Boundedness of estimating function  $g$  is a characterizing feature of robust M-estimators. Standard block resampling approaches need to solve equation  $\psi_k(X_{(k)}^*, \hat{\theta}_k^*) = 0$  for each bootstrap ( $k = n$ ) or subsampling ( $k = m < n$ ) random sample  $X_{(k)}^*$ , which is computationally demanding. Instead, we consider the following Taylor expansion of (4) around the true parameter  $\theta_0$ :

$$\hat{\theta}_n - \theta_0 = -[\nabla_{\theta} \psi_n(X_{(n)}, \theta_0)]^{-1} \psi_n(X_{(n)}, \theta_0) + o_p(1), \quad (5)$$

where  $\nabla_{\theta} \psi_n(X_{(n)}, \theta_0)$  denotes the derivative of function  $\psi_n$  with respect to  $\theta$ . Based on this expansion, we use  $-[\nabla_{\theta} \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} \psi_k(X_{(k)}^*, \hat{\theta}_n)$  as an approximation of  $\hat{\theta}_k^* - \hat{\theta}_n$  in the definition of the resampling scheme estimating the sampling distribution of  $\hat{\theta}_n - \theta_0$ . This avoids computing  $\hat{\theta}_k^*$  and  $[\nabla_{\theta} \psi_n(X_{(k)}^*, \hat{\theta}_n)]^{-1}$  on resampled data.

Given a normalization constant  $\tau_n$ , a robust fast resampling distribution for  $\tau_n(\hat{\theta}_n - \theta_0)$  is defined by

$$L_{n,m}^{RF*}(x) = \frac{1}{N} \sum_{s=1}^N \mathbb{I} \left( \tau_k \left( -[\nabla_{\theta} \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} \psi_k(X_{(k),s}^*, \hat{\theta}_n) \right) \leq x \right), \quad (6)$$

where  $\mathbb{I}(\cdot)$  is the indicator function and  $s$  indexes the  $N$  possible random sam-

ples generated by subsampling and bootstrap procedures, respectively. The main assumptions under which the fast resampling distribution (7) consistently estimates the unknown sampling distribution of  $\tau_n(\hat{\theta}_n - \theta_0)$  in a time series context are given, e.g., in Hong and Scaillet (2006) for the subsampling (Assumption 1) and in Goncalves and White (2004) for the bootstrap (Assumption A and Assumptions 2.1 and 2.2).

Before analyzing the robustness properties of the robust fast resampling distribution (7), we provide a final remark on the rate of convergence. We denote by  $E^*$  the expectation with respect to the probability measure induced by the resampling method. As pointed out for instance in Hall et al. (1995), the overlapping scheme of classical resampling methods generally implies  $E^*[\hat{\theta}_n^*] \neq E^*[\hat{\theta}_n]$ . Because of this distortion, the rate of convergence of the resampling distribution decreases. To overcome the problem, a simple solution consists in considering the recentered statistic  $\hat{\theta}_n^* - E^*[\hat{\theta}_n^*]$ , instead of  $\hat{\theta}_n^* - \hat{\theta}_n$ . Note that the recentering approach can be applied with our robust fast procedure as well. Indeed, as showed in Andrews (2002) in relation to a fast bootstrap method, the robust fast recentered distribution

$$L_{n,m}^{RF*}(x) = \frac{1}{N} \sum_{s=1}^N \mathbb{I} \left( \tau_k \left( - [\nabla_{\theta} \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} (\psi_k(X_{(k),s}^*, \hat{\theta}_n) - E^*[\psi_k(X_{(k),s}^*, \hat{\theta}_n)]) \right) \leq x \right), \quad (7)$$

implies by construction the nondistortion condition

$$E^* \left[ [\nabla_{\theta} \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} (\psi_k(X_{(k)}^*, \hat{\theta}_n) - E^*[\psi_k(X_{(k)}^*, \hat{\theta}_n)]) \right] = 0. \quad (8)$$

Using our robust approach the computation of  $E^*[\psi_k(X_{(k)}^*, \hat{\theta}_n)]$  only marginally increases the computational costs, which are instead very large for the computation of  $E^*[\hat{\theta}_n^*]$  with the classical methods. Unreported Monte Carlo simu-

lations show a better accuracy of the recentered procedures for both standard and robust approaches. In particular, the improvement produced by the recentering is more evident for nonsymmetric confidence intervals, while it is less pronounced for symmetric two-sided confidence intervals.

### 3.2 Robust Resampling Methods and Quantile Breakdown Point

In the computation of (7) we only need point estimates for  $\theta_0$  and the matrix  $-\nabla_{\theta}\psi_n(X_{(n)}, \theta_0)]^{-1}$ , based on the whole sample  $X_{(n)}$ . These estimates are given by  $\hat{\theta}_n$  and  $-\nabla_{\theta}\psi_n(X_{(n)}, \hat{\theta}_n)]^{-1}$ , respectively. Thus, a computationally very fast procedure is obtained. This feature is not shared by standard resampling schemes, which can easily become unfeasible when applied to robust statistics.

A closer look at  $-\nabla_{\theta}\psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} \psi_k(X_{(k),s}^*, \hat{\theta}_n)$  reveals that this quantity can degenerate to infinity when (i) the matrix  $\nabla_{\theta}\psi_n(X_{(n)}, \hat{\theta}_n)$  is singular or (ii) the estimating function is not bounded. Since we are making use of a robust (bounded) estimating function  $g$  situation (ii) cannot arise. From these arguments, we obtain the following corollary.

**Corollary 4** *Let  $b$  be the breakdown point of the robust M-estimator  $\hat{\theta}_n$  defined by (4). The  $t$ -quantile breakdown point of resampling distribution (7) is given by  $b_t = \min(b, b_{\nabla\psi})$ , where*

$$b_{\nabla\psi} = \frac{1}{n} \cdot \inf_{1 \leq p \leq \lceil n/2 \rceil} \left\{ p \mid \text{there exists } Z_p^\zeta \in Z_p^\zeta \text{ such that } \det(\nabla_{\theta}\psi_n(X_{(n)} + Z_p^\zeta, \hat{\theta}_n)) = 0 \right\}. \quad (9)$$

The quantile breakdown point of our robust fast resampling distribution is the minimum of the breakdown point of M-estimator  $\hat{\theta}_n$  and matrix  $\nabla_{\theta}\psi_n(X_{(n)}, \hat{\theta}_n)$ .



In particular, if  $b_{\nabla\psi} \geq b$ , the quantile breakdown point of our robust resampling distribution (7) is maximal, independent of confidence level  $t$ .

Finally, we consider a last remark on our robust fast resampling approach. Unreported Monte Carlo simulations show that the application of our fast approach to a nonrobust estimating function only marginally provides some robustness improvements. It turns out that in order to ensure robustness both the fast approach and a robust bounded estimating function are necessary.

## 4 Breakdown Point and Data Driven Choice of the Block Size

A main issue in the application of block resampling procedures is the choice of the block size  $m$ , since accuracy of the resampling distribution depends strongly on this parameter (see, e.g., Lahiri, 2001). In this section, we study the robustness of data driven block size selection approaches for subsampling and bootstrap procedures. We first consider the MCIV and CM proposed in Romano and Wolf (2001) for the subsampling. In a second step, we analyze the HHJ method for the bootstrap. For these methods, we characterize the smallest fraction of outliers in the original sample such that the data driven choice of the block size fails and diverges to infinity. For brevity, we denote by  $m_u(X_{(n)})$ ,  $u = \text{MCIV, CM, HHJ}$ , the block size choice implied by each of these methods. By definition, the breakdown point of  $m_u$  is defined by

$$b_t^u = \frac{1}{n} \cdot \inf_{1 \leq p \leq \lceil n/2 \rceil} \{p \mid \text{there exists } Z_p^\zeta \in \mathcal{Z}_p^\zeta \text{ such that } m_u(X_{(n)} + Z_p^\zeta) = \infty\}. \quad (10)$$

## 4.1 Subsampling

Denote by  $b_t^{OS,J}$ ,  $J = MCIV, CM$ , the breakdown point of the overlapping subsampling based on the MCIV and CM methods, respectively.

### 4.1.1 Minimum Confidence Index Volatility

A consistent method for a data driven choice of the block size  $m$  is based on the minimization of the confidence interval volatility index across the admissible values of  $m$ . For brevity, we present the method for one-sided confidence intervals. Modifications for two-sided confidence intervals are obvious.

**Definition 5** Let  $m_{\min} < m_{\max}$  and  $k \in \mathbb{N}$  be fixed. For  $m \in \{m_{\min} - k, \dots, m_{\max} + k\}$ , denote by  $Q_t^*(m)$  the  $t$ -subsampling quantile for the block size  $m$ . Further, let  $\overline{Q}_t^{*k}(m)$  be the average quantile  $\overline{Q}_t^{*k}(m) := \frac{1}{2k+1} \sum_{i=-k}^{i=k} Q_t^*(m+i)$ . The confidence interval volatility (CIV) index is defined for  $m \in \{m_{\min}, \dots, m_{\max}\}$  by

$$CIV(m) := \frac{1}{2k+1} \sum_{i=-k}^{i=k} \left( Q_t^*(m+i) - \overline{Q}_t^{*k}(m) \right)^2. \quad (11)$$

Let  $\mathcal{M} := \{m_{\min}, \dots, m_{\max}\}$ . The data driven block size that minimizes the confidence interval volatility index is

$$m_{MCIV} = \arg \inf_{m \in \mathcal{M}} \{CIV(m) : CIV(m) \in \mathbb{R}^+\}, \quad (12)$$

where, by definition,  $\arg \inf(\emptyset) := \infty$ .

The block size  $m_{MCIV}$  minimizes the empirical variance of the upper bound in a subsampling confidence interval with nominal confidence level  $t$ . Using

Theorem 2, the formula for the breakdown point of  $m_{MCIV}$  is given in the next corollary.

**Corollary 6** *Let  $b$  be the breakdown point of estimator  $\hat{\theta}_n$ . For given  $t \in (0, 1)$ , let  $b_t^{OS}(m)$  be the overlapping subsampling upper  $t$ -quantile breakdown point in Theorem 2, as a function of the block size  $m \in \mathcal{M}$ . It then follows:*

$$b_t^{OS, MCIV} \leq \sup_{m \in \mathcal{M}} \inf_{j \in \{-k, \dots, k\}} b_t^{OS}(m + j). \quad (13)$$

The dependence of the breakdown point formula for the MCIV on the breakdown point of subsampling quantiles is identical to the iid case. However, the much smaller quantile breakdown points in the time series case make the data driven choice  $m_{MCIV}$  very unreliable in presence of outliers. For instance, for the block size  $n = 120$  and a maximal breakdown point statistic such that  $b = 0.5$ , the breakdown point of MCIV for  $t = 0.95$  is less than 0.05, i.e., just 6 outliers are sufficient to break down the MCIV data driven choice of  $m$ . For the same sample size, the breakdown point of the MCIV method is larger than 0.3 in the iid case.

#### 4.1.2 Calibration Method

Another consistent method for a data driven choice of the block size  $m$  can be based on a calibration procedure in the spirit of Loh (1987). Again, we present this method for the case of one-sided confidence intervals only. The modifications for two-sided confidence intervals are straightforward.

**Definition 7** *Fix  $t \in (0, 1)$  and let  $(X_1^*, \dots, X_n^*)$  be a nonoverlapping moving block bootstrap sample generated from  $X_{(n)}$  with block size  $m$ . For each*

bootstrap sample, denote by  $Q_t^{**}(m)$  the  $t$ -subsampling quantile according to block size  $m$ . The data driven block size according to the calibration method is defined by

$$m_{CM} := \arg \inf_{m \in \mathcal{M}} \{ |t - P^* [\hat{\theta}_n \leq Q_t^{**}(m)]| : P^* [Q_t^{**}(m) \in \mathbb{R}] > 1 - t \}, \quad (14)$$

where, by definition,  $\arg \inf(\emptyset) := \infty$ , and  $P^*$  is the nonoverlapping moving block bootstrap probability distribution.

In the approximation of the unknown underlying data generating mechanism in Definition 7, we use a nonoverlapping moving block bootstrap for ease of exposition. It is possible to consider also other resampling methods; see, e.g., Romano and Wolf (2001). By definition,  $m_{CM}$  is the block size for which the bootstrap probability of the event  $[\hat{\theta}_n \leq Q_t^{**}(m)]$  is as near as possible to the nominal level  $t$  of the confidence interval, but which at the same time ensures that the resampling quantile breakdown probability of the calibration method is less than  $t$ . The last condition is necessary to ensure that the calibrated block size  $m_{CM}$  does not imply a degenerate subsampling quantile  $Q_t^{**}(m_{CM})$  with a too large probability.

**Corollary 8** *Let  $b$  be the breakdown point of estimator  $\hat{\theta}_n$ ,  $t \in (0, 1)$ , and define:*

$$b_t^{OS**}(m) \leq \frac{1}{n} \cdot \left[ \inf_{q \in \mathbb{N}, q \leq r} \left\{ p = \lceil mb \rceil \cdot q \left| P \left( \text{BIN} \left( r, \frac{q}{r} \right) < Q^{OS} \right) < 1 - t \right\} \right],$$

where  $Q^{OS} = \frac{\lceil (n-m+1)(1-t) \rceil + \lceil mb \rceil - 1}{m}$ . It then follows:

$$b_t^{OS,CM} \leq \sup_{m \in \mathcal{M}} \{ b_t^{OS**}(m) \}. \quad (15)$$

Because of the use of the moving block bootstrap instead of the standard iid bootstrap in the CM for time series, equation (15) is quite different from the formula for the iid case in Camponovo, Scaillet and Trojani (2009). Similar to the iid case, the theoretical results in Table 2 and the Monte Carlo results in the last section of this paper indicate a higher stability and robustness of the CM relative to the MCIV method. Therefore, from a robustness perspective, the former should be preferred when consistent bootstrap methods are available. As discussed in Romano and Wolf (2001), the application of the calibration method in some settings can be computationally expensive. In contrast to our fast robust resampling approach, a direct application of the subsampling to robust estimators can easily become computationally prohibitive in combination with the CM.

## 4.2 Moving Block Bootstrap

The data driven method for the block size selection in Hall, Horowitz and Jing (1995) first computes the optimal block size for a subsample of size  $m < n$ . In a second step it uses Richardson extrapolation in order to determine the optimal block size for the whole sample.

**Definition 9** *Let  $m < n$  be fixed and split the original sample in  $n - m + 1$  overlapping blocks of size  $m$ . Fix  $l_{\min} < l_{\max} < m$  and for  $l \in \{l_{\min}, \dots, l_{\max}\}$  denote by  $Q_t^*(m, l, i)$  the  $t$ -moving block bootstrap quantile computed with the block size  $l$  using the bootstrap  $m$ -block  $(X_i, \dots, X_{i+m-1})$ ,  $1 \leq i \leq n - m + 1$ .  $\overline{Q}_t^*(m, l) := \frac{1}{n-m+1} \sum_{i=1}^{n-m+1} Q_t^*(m, l, i)$  is the corresponding average quantile. Finally, denote by  $Q_t^*(n, l')$  the  $t$ -moving block bootstrap quantile computed with block size  $l' < n$  based on the original sample  $X_{(n)}$ . For  $l \in \{l_{\min}, \dots, l_{\max}\}$*

define the MSE index is defined as

$$MSE(l) := \left( \overline{Q}_t^*(m, l) - Q_t^*(n, l') \right)^2 + \frac{1}{n - m + 1} \sum_{i=1}^{n-m+1} (Q_t^*(m, l, i) - Q_t^*(n, l'))^2, \quad (16)$$

and set:

$$l_{HHJ} = \arg \inf_{l \in \{l_{\min}, \dots, l_{\max}\}} \{MSE(l) : MSE(l) \in \mathbb{R}^+\}, \quad (17)$$

where, by definition,  $\arg \inf(\emptyset) := \infty$ . The optimal block size for the whole  $n$ -sample is defined by

$$m_{HHJ} = l_{HHJ} \left( \frac{n}{m} \right)^{1/5}. \quad (18)$$

As discussed in Bühlmann and Künsch (1999), the HHJ method is not fully data driven, because it is based on some starting parameter values  $m$  and  $l'$ . However, the algorithm can be iterated. After computing the first value  $m_{HHJ}$ , we can set  $l' = m_{HHJ}$  and iterate the same procedure. As pointed out in Hall, Horowitz and Jing (1995), this procedure often converges in one step. Also for this data-driven method, the application of the classical bootstrap approach to robust estimators easily becomes computationally unfeasible.

**Corollary 10** *Let  $b$  be the breakdown point of estimator  $\hat{\theta}_n$ . For given  $t \in (0, 1)$ , let  $b_t^{NB, m}(l)$  and  $b_t^{OB, m}(l)$  be the nonoverlapping and overlapping moving block upper  $t$ -quantile breakdown point in Theorem 2, as a function of the block size  $l \in \{l_{\min}, \dots, l_{\max}\}$  and a size  $m$  of the initial sample. It then follows for  $K = NS, OS$ :*

$$b_t^{K, HHJ} \leq \frac{m}{n} \cdot \sup_{l \in \{l_{\min}, \dots, l_{\max}\}} b_t^{K, m}(l). \quad (19)$$

The computation of the optimal block size  $l_{HHJ}$  based on smaller subsamples of size  $l \ll m < n$ , causes a large instability in the computation of  $m_{HHJ}$ .

Because of this effect, the MSE index in (16) can easily deteriorate even with a small contamination. Indeed, it is enough that the computation of the quantile degenerates just in a single  $m$ -block in order to imply a degenerated MSE. Table 2 confirms this intuition. For  $n = 120$ ,  $b = 0.5$ , and  $t = 0.95$ , the upper bound on the breakdown point of the HHJ method is half that of the CM, even if for small block sizes the quantile breakdown point of subsampling procedures is typically lower than that of bootstrap methods.

## 5 Monte Carlo Simulations and Empirical Application

We compare through Monte Carlo simulations the accuracy of classical resampling procedures and our fast robust approach in estimating the confidence interval of the autoregressive parameter in a linear AR(1). Moreover, as a final exercise, we consider an application to real data by testing the predictability of future stock returns with the classic and our robust fast subsampling.

### 5.1 AR(1) Model

Consider the linear AR(1) model of the form:

$$X_t = \theta X_{t-1} + \epsilon_t, \quad X_0 \sim N\left(0, \frac{1}{1 - \theta^2}\right), \quad (20)$$

where  $|\theta| < 1$  and  $\{\epsilon_t\}$  is a sequence of iid standard normal innovations. We denote by  $\hat{\theta}_n^{OLS}$  the (nonrobust) OLS estimator of  $\theta_0$ , which is the solution of

equation:

$$\psi_n^{OLS}(X_{(n)}, \hat{\theta}_n^{OLS}) := \frac{1}{n-1} \sum_{t=2}^n X_{t-1}(X_t - \hat{\theta}_n^{OLS} X_{t-1}) = 0. \quad (21)$$

To apply our robust fast resampling approach, we consider a robust estimator  $\hat{\theta}_n^{ROB}$  defined by

$$\psi_n^{ROB}(X_{(n)}, \hat{\theta}_n^{ROB}) := \frac{1}{n-1} \sum_{t=2}^n h_c(X_{t-1}(X_t - \hat{\theta}_n^{ROB} X_{t-1})) = 0, \quad (22)$$

where  $h_c(x) := x \cdot \min(1, c/|x|)$ ,  $c > 1$ , is the Huber function; see Künsch (1984).

To study the robustness of the different resampling methods under investigation, we consider for brevity replacement outliers random samples  $(\tilde{X}_1, \dots, \tilde{X}_n)$  generated according to

$$\tilde{X}_t = (1 - p_t)X_t + p_t \cdot X_{1.5max}, \quad (23)$$

where  $X_{1.5max} = 1.5 \cdot \max(X_1, \dots, X_n)$  and  $p_t$  is an iid 0 – 1 random sequence, independent of process (20) and such that  $P[p_t = 1] = \eta$ ; see Martin and Yohai (1986). The probability of contamination is set to  $\eta = 1.5\%$ , which is a very small contamination of the original sample.

### 5.1.1 The Standard Strictly Stationary Case

We construct symmetric resampling confidence intervals for the true parameter  $\theta_0$ . Hall (1988) and more recent contributions, as for instance Politis, Romano and Wolf (1999), highlight a better accuracy of symmetric confidence intervals, which even in asymmetric settings can be shorter than asymmetric



confidence intervals. Andrews and Guggenberger (2009, 2010a) and Mikusheva (2007) also show that because of a lack of uniformity in pointwise asymptotics, nonsymmetric subsampling confidence intervals for autoregressive models can imply a distorted asymptotic size, which is instead correct for symmetric confidence intervals.

Using OLS estimator (21), we compute both overlapping subsampling and moving block bootstrap distributions for the distribution of  $\sqrt{n}|\hat{\theta}_n^{OLS} - \theta_0|$ . Using robust estimator (22), we compute overlapping robust fast subsampling and moving block bootstrap distributions for the distribution of  $\sqrt{n}|\hat{\theta}_n^{ROB} - \theta_0|$ . Standard resampling methods combined with data driven block size selection methods for robust estimator (22) are computationally too expensive.

We generate  $N = 1000$  samples of size  $n = 180$  according to model (20) for the parameter choices  $\theta_0 = 0.5, 0.6, 0.7, 0.8$ . We select the subsampling block size using MCIV and CM for  $\mathcal{M} = \{9, 10, 12, 15, 18\}$ . For the bootstrap, we apply HHJ method with  $l' = 12$ ,  $m = 30$ ,  $l_{min} = 6$ , and  $l_{max} = 10$ . The degree of robustness is  $c = 5$ ; the significance level is  $1 - \alpha = 0.95$ .

We first analyze the finite sample coverage and the power of resampling procedures in a test of the null hypothesis  $\mathcal{H}_0 : \theta_0 = 0.5$ . Figure 1 plots the empirical frequencies of rejection of the null hypothesis  $\mathcal{H}_0 : \theta_0 = 0.5$  for  $\theta_0 = 0.5$  and different values of the alternative hypothesis:  $\theta_0 \in \{0.6, 0.7, 0.8\}$ .

Without contamination (left column,  $\eta = 0\%$ ), we find that our robust fast approach and the classical procedures provide accurate and comparable results. In particular, when  $\theta_0 = 0.5$ , the size values for the classical moving block bootstrap and subsampling with CM are 0.045 and 0.056, respectively. With our robust approach, for the robust fast bootstrap and robust fast subsampling with CM we obtain 0.055 and 0.061, which both imply size values very close to

the nominal level  $\alpha = 0.05$ . For the robust fast subsampling and the classical subsampling with MCIV the size is larger than 0.067, which suggests a lower accuracy of the MCIV relative to the CM. When  $\theta_0 \neq 0.5$ , the proportion of rejections of our robust fast approach remains larger than that of the classical methods. For instance, when  $\theta_0 = 0.7$ , this difference in power between robust fast subsampling and subsampling with CM is close to 10%. It is even larger than 10% in a comparison between the moving block robust fast bootstrap and the classical bootstrap.

If we consider the contaminated Monte Carlo simulations (right column,  $\eta = 1.5\%$ ), the size increases for  $\theta_0 = 0.5$  for nonrobust methods, which are found to be dramatically oversized. In the case of nonrobust subsampling methods the size is even larger than 0.3. In contrast, the size of our robust fast approach remains closer to the nominal level  $\alpha = 0.05$ . In particular, the size is 0.082 for the robust fast subsampling with CM. A contamination tremendously deteriorates also the power of nonrobust methods. As  $\theta_0$  increases, we find that the power curve of nonrobust methods is not monotonically increasing, with low frequencies of rejection even when  $\theta_0$  is far from 0.5. For instance, for  $\theta_0 = 0.8$ , the power of nonrobust methods is close to 50%, but that of our robust approach is larger than 90%.

In a second exercise, we examine the sensitivity of the different resampling procedures with respect to a single point contamination of the original sample. For each Monte Carlo sample, let:

$$X_{\max} = \arg \max_{X_1, \dots, X_n} \{u(X_i) | u(X_i) = X_i - \theta X_{i-1}, \text{ under } \mathcal{H}_0\}. \quad (24)$$

We modify  $X_{\max}$  over a grid  $\{X_{\max} + i; i = 1, 2, 3, 4\}$ . Then, we analyze the

sensitivity of the resulting empirical averages of  $p$ -values for testing the null hypothesis  $\mathcal{H}_0 : \theta_0 = 0.5$ . In Figure 2, we plot the resulting empirical  $p$ -values. As expected, our robust fast approach shows a desirable stability for both subsampling and bootstrap methods.

### 5.1.2 The Near-to-Unit-Root Case

As a second application, we consider the near-to-unit-root case. Moving block bootstrap procedures are then inconsistent, but the studentized subsampling based on symmetric confidence intervals is consistent; see Andrews and Guggenberger (2009, 2010b), and Mikusheva (2007). Therefore, we focus exclusively on the latter method.

Consider the OLS estimator (21) and denote by  $\hat{\sigma}_n^{OLS}$  the estimated standard deviation of  $\hat{\theta}_n^{OLS}$ . The studentized subsampling approximates the distribution of  $|\hat{\theta}_n^{OLS} - \theta_0|/\hat{\sigma}_n^{OLS}$  by the empirical distribution of  $|\hat{\theta}_m^{*OLS} - \hat{\theta}_n^{OLS}|/\hat{\sigma}_m^{*OLS}$ , where  $\hat{\sigma}_m^{*OLS}$  denotes the estimated standard deviation of  $\hat{\theta}_m^{*OLS}$  based on the subsampling block. Let  $\hat{\sigma}_n^{ROB}$  be the estimated standard deviation of  $\hat{\theta}_n^{ROB}$ . Using our robust approach, the robust fast subsampling approximates the distribution of  $|\hat{\theta}_n^{ROB} - \theta_0|/\hat{\sigma}_n^{ROB}$  by the empirical distribution of the term  $|(-[\nabla_{\theta}\psi_m^{ROB}(X_{(m)}^*, \hat{\theta}_n^{ROB})]^{-1}\psi_m^{ROB}(X_{(m)}^*, \hat{\theta}_n^{ROB}))|/\hat{\sigma}_n^{*ROB}$ , where  $\hat{\sigma}_n^{*ROB}$  denotes the estimated standard deviation of  $\hat{\theta}_m^{*ROB}$ ; see also Hong and Scaillet (2006).

We generate  $N=1000$  samples of size  $n = 180$  according to model (20) for the parameter choices  $\theta_0 = 0.8, 0.85, 0.9, 0.95$ , and simulate contaminated samples  $(\tilde{X}_1, \dots, \tilde{X}_n)$  according to (28) as before. Since bootstrap methods are inconsistent, for the selection of the block size only MCIV is recommended. However, in this setting Romano and Wolf (2001) obtain accurate results even with the subsampling based on CM. Consequently, for comparison purposes in

our experiments we consider both MCIV and CM with  $\mathcal{M} = \{9, 10, 12, 15, 18\}$ . The degree of robustness is  $c = 9$ .

We analyze the finite sample size and the power of resampling procedures in a test of the null hypothesis  $\mathcal{H}_0 : \theta_0 = 0.8$ . The significance level is  $1 - \alpha = 0.95$ . Figure 3 plots the empirical frequencies of rejection of the null hypothesis  $\mathcal{H}_0 : \theta_0 = 0.8$  for  $\theta_0 = 0.8$  and different values  $\theta_0 \in \{0.85, 0.9, 0.95\}$  of the alternative hypothesis. As in the previous Monte Carlo setting, we find that without contamination (left column,  $\eta = 0\%$ ) our robust fast approach and the classical procedures yield accurate and comparable results. When  $\theta_0 = 0.8$ , the difference between the nominal level  $\alpha = 0.05$  and the size of all methods under investigation is less than 1.3%. For large  $\theta_0$ , we find that the power of the robust fast subsampling is higher. The difference in power between robust and nonrobust methods is near to 10% for  $\theta_0 = 0.9$  and for both data driven choices of the block size. When we consider the contaminated Monte Carlo simulations (right column,  $\eta = 1.5\%$ ) the size of the robust fast subsampling with MCIV and CM (0.086 and 0.057, respectively) is slightly closer to the nominal level than that of the subsampling (0.103 with MCIV and 0.096 with CM) for  $\theta_0 = 0.8$ . More strikingly, we also find that a contamination by outliers tremendously deteriorates the power of the subsampling approach. As  $\theta_0$  increases towards the boundary value 1, the power curve of the subsampling is nonmonotonic, with frequencies of rejection less than 20% even when  $\theta_0 = 0.95$  for both MCIV and CM. In contrast, the power of the robust fast subsampling is substantial and larger than 80% for  $\theta_0 = 0.95$ .

## 5.2 Stock Return Predictability

Consider the predictive regression model:

$$y_t = \alpha + \beta x_{t-1} + \epsilon_t, \quad (25)$$

where, for  $t = 1, \dots, n$ ,  $\{y_t\}$  denotes the stock return,  $\{x_t\}$  denotes the explanatory variable and  $\{\epsilon_t\}$  is the error term. We use the subscript 0 to indicate the true value  $\beta_0$  of the parameter  $\beta$ .

Recently, several testing procedures have been proposed in order to test the nonpredictability hypothesis  $\mathcal{H}_0 : \beta_0 = 0$ ; see among others Campbell and Yogo (2006), Jansson and Moreira (2006), and Amihud, Hurvich and Wang (2008). Indeed, because of the endogeneity of the explanatory variables in this setting, classic asymptotic theory based on OLS estimator becomes inaccurate. Moreover, as emphasized in Torous, Valkanov and Yan (2004), various state variables considered as predictors follow a nearly integrated process, which complicates inference on parameter  $\beta$ . As advocated, e.g., in Wolf (2000), the subsampling approach can be applied for testing the hypothesis of no predictability.

In this study, we analyze the predictive power of dividend yields for stock returns with the classic studentized subsampling and our robust approach. We define the one-period real total return as

$$R_t = (P_t + d_t)/P_{t-1}, \quad (26)$$

where  $P_t$  is the end of month real stock price and  $d_t$  is the real dividend paid

during month  $t$ . Furthermore, we define the annualized dividend series  $D_t$  as

$$D_t = d_t + (1 + r_t)d_{t-1} + (1 + r_t)(1 + r_{t-1})d_{t-2} + \cdots + (1 + r_t) \cdots (1 + r_{t-10})d_{t-11}, \quad (27)$$

where  $r_t$  is the one-month treasury-bill rate. Finally, we set  $y_t = \ln(R_t)$  and  $x_t = D_t/P_t$  as in Wolf (2000).

We implement the studentized classic subsampling and our robust approach based on MCIV and CM on US equity data. We consider monthly S&P 500 index data from Shiller (2000). In Figure 4, we plot the dividend yield and the log of return of the S&P 500 for the period 1964-2009. The nearly integrated features of the dividend yield are well-known and apparent in the top panel of Figure 4. We test the nonpredictability hypothesis  $\mathcal{H}_0 : \beta_0 = 0$  for the whole period 1964-2009, consisting of 540 observations, and the subperiods 1964-1993 and 1994-2009, consisting of 360 and 180 observations, respectively. For each method and period, we construct 95% confidence intervals for parameter  $\beta$ . Table 3 reports our empirical results.

In the whole period 1964-2009 and in the subperiod 1964-1993, classic and robust subsampling with MCIV and CM provide very similar confidence intervals. In both these periods, the procedures under investigation provide also evidence in favor of predictability. However, it is interesting to note that our robust approach implies smaller confidence intervals than the classic method, leading to a stronger rejection of the null of no predictability. Finally, in the subperiod 1994-2009, only our robust approach produces significant evidence of predictability. The classic subsampling with MCIV and CM implies extremely large confidence intervals which lead to a nonrejection of  $\mathcal{H}_0$ . This finding seems to confirm the robustness problem of the classic approach in our

Monte Carlo simulations. Indeed, the subperiod 1994-2009 and in particular the year 2009 is characterized by a large proportion of unusual observations. For instance, we find that observation for the return October 2008 is the most influential data point for the whole period 1964-2009. As shown in the previous section, the presence of anomalous observations may dramatically deteriorate the performance of nonrobust resampling methods. Consequently, the nonrejection of  $\mathcal{H}_0$  caused by the large confidence intervals provided by the classic subsampling suggests a low power of this approach in these cases.

## 6 Conclusions

Theoretical breakdown point formulas and Monte Carlo evidence highlight a dramatic unexpected lack of robustness of classical block resampling methods for time series. This problem affects block bootstrap and subsampling procedures as well, and it is much worse than a related problem analyzed recently by the literature in the iid context. To overcome the problem, we propose a general robust fast resampling approach, which is applicable to a wide class of block resampling methods, and show that it implies good theoretical quantile breakdown point properties. In the context of a simple linear AR(1) model, our Monte Carlo simulations show that the robust resampling delivers more accurate and efficient results, in some cases to a dramatic degree, than other standard block resampling schemes in presence and absence of outliers in the original data. A real data application to testing for stock returns predictability provides more consistent evidence in favor of the predictability hypothesis using our robust resampling approach.

## Appendix: Proofs

**Proof of Theorem 2.** The value  $\frac{\lceil mb \rceil}{n}$  is the minimal fraction of outliers, which is necessary to cause the breakdown of statistic  $T$  in a block of size  $m$ . Therefore, the first inequality is satisfied.

For the second inequality, we denote by  $X_{(m),i}^N = (X_{(i-1)m+1}, \dots, X_{im})$ ,  $i = 1, \dots, r$  and  $X_{(m),i}^O = (X_i, \dots, X_{i+m-1})$ ,  $i = 1, \dots, n-m+1$ , the nonoverlapping and overlapping blocks of size  $m$ , respectively. Given the original sample  $X_{(n)}$ , for the first nonoverlapping block  $X_{(m),1}^N$ , consider the following type of contamination:

$$X_{(m),1}^N = (X_1, \dots, X_{m-\lceil mb \rceil}, Z_{m-\lceil mb \rceil+1}, \dots, Z_m), \quad (28)$$

where  $X_i$ ,  $i = 1, \dots, m - \lceil mb \rceil$  and  $Z_j$ ,  $j = m - \lceil mb \rceil + 1, \dots, m$ , denote the noncontaminated and contaminated points, respectively. By construction, the first  $m - \lceil mb \rceil + 1$  overlapping blocks  $X_{(m),i}^O$ ,  $i = 1, \dots, m - \lceil mb \rceil + 1$ , contain  $\lceil mb \rceil$  outliers. Consequently,  $T(X_{(m),i}^O) = +\infty$ ,  $i = 1, \dots, m - \lceil mb \rceil + 1$ . Assume that the first  $p < r - 1$  nonoverlapping blocks  $X_{(m),i}^N$ ,  $i = 1, \dots, p$ , have the same contamination as in (28). Because of this contamination, the number of statistics  $T_{n,m}^*$  which diverge to infinity is  $mp - \lceil mb \rceil + 1$ .

$Q_t^* = +\infty$ , when the proportion of statistics  $T_{n,m}^*$  with  $T_{n,m}^* = +\infty$  is larger than  $(1 - t)$ . Therefore,  $b_t^{OS} \leq \inf_{\{p \in \mathbb{N}, p \leq r-1\}} \left\{ p \cdot \frac{\lceil mb \rceil}{n} \left| \frac{mp - \lceil mb \rceil + 1}{n - m + 1} > 1 - t \right. \right\}$ . ■

**Proof of Theorem 3.** The proof of the first inequalities follows straightforward from the proof of the first inequality in Theorem 2. We focus on the



second inequalities.

*Case (i): Nonoverlapping Moving Block Bootstrap.* Consider  $X_{(m),i}^N$ ,  $i = 1, \dots, r$ . Assume that  $p_2$  of these nonoverlapping blocks are contaminated with exactly  $p_1$  outliers for each block, while the remaining  $(r - p_2)$  are noncontaminated (0 outlier), where  $p_1, p_2 \in \mathbb{N}$  and  $p_1 \leq m$ ,  $p_2 \leq r - 1$ . The nonoverlapping moving block bootstrap constructs a  $n$ -sample randomly selecting with replacement  $r$  nonoverlapping blocks. Let  $X$  be the random variable which denotes the number of contaminated blocks in the random bootstrap sample. It follows that  $X \sim \text{BIN}(r, \frac{p_2}{r})$ .

By Definition 1,  $Q_t^* = +\infty$ , when the proportion of statistics  $T_{n,n}^*$  with  $T_{n,n}^* = +\infty$  is larger than  $(1 - t)$ . The smallest number of outliers such that  $T_{n,n}^* = +\infty$  is by definition  $nb$ . Let  $p_1, p_2 \in \mathbb{N}$ ,  $p_1 \leq m$ ,  $p_2 \leq r - 1$ . Consequently,

$$b_t^{NB} \leq \frac{1}{n} \cdot \left[ \inf_{\{p_1, p_2 \in \mathbb{N}\}} \left\{ p = p_1 \cdot p_2 \left| P\left( \text{BIN}\left(r, \frac{p_2}{r}\right) > \frac{nb}{p_1} \right) > 1 - t \right\} \right].$$

*Case (ii): Overlapping Moving Block Bootstrap.* Given the original sample  $X_{(n)}$ , consider the same nonoverlapping blocks as in (i), where the contamination of the  $p_2$  contaminated blocks has the structure defined in (28). The overlapping moving block bootstrap constructs a  $n$ -sample randomly selecting with replacement  $r$  overlapping blocks of size  $m$ . Let  $X$  be the random variable which denotes the number of contaminated blocks in the random bootstrap sample. It follows that  $X \sim \text{BIN}(r, \frac{mp_2 - p_1 + 1}{n - m + 1})$ .

By Definition 1,  $Q_t^* = +\infty$ , when the proportion of statistics  $T_{n,n}^*$  with  $T_{n,n}^* = +\infty$  is larger than  $(1 - t)$ . The smallest number of outliers such that  $T_{n,n}^* = +\infty$  is by definition  $nb$ . Let  $p_1, p_2 \in \mathbb{N}$ ,  $p_1 \leq m$ ,  $p_2 \leq r - 1$ . Consequently,

$$b_t^{OB} \leq \frac{1}{n} \cdot \left[ \inf_{\{p_1, p_2\}} \left\{ p = p_1 \cdot p_2 \left| P \left( \text{BIN} \left( r, \frac{mp_2 - p_1 + 1}{n - m + 1} \right) > \frac{nb}{p_1} \right) > 1 - t \right\} \right]. \quad \blacksquare$$

**Proof of Corollary 4.** Consider the robust fast approximation of  $(\hat{\theta}_k^* - \hat{\theta}_n)$  given by

$$-[\nabla_{\theta} \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} \psi_k(X_{(k),s}^*, \hat{\theta}_n), \quad (29)$$

where  $k = n$  or  $k = m$ . Assuming a bounded estimating function, expression (29) may degenerate only when, (i)  $\hat{\theta}_n \notin \mathbb{R}$  or (ii) the matrix  $[\nabla_{\theta} \psi_n(X_{(n)}, \hat{\theta}_n)]$  is singular, i.e.  $\det([\nabla_{\theta} \psi_n(X_{(n)}, \hat{\theta}_n)]) = 0$ . If (i) and (ii) are not satisfied, it turns out that the quantile  $Q_t^*$  is bounded,  $\forall t \in (0, 1)$ . Let  $b$  be the breakdown point of  $\hat{\theta}_n$  and  $b_{\nabla \psi}$  be the smallest fraction of outliers in the original sample such that condition (ii) is satisfied, the breakdown point of  $Q_t^*$  is given by  $b_t = \min(b, b_{\nabla \psi})$ .  $\blacksquare$

**Proof of Corollary 6.** Denote  $b_t^{OS}(m)$ , the overlapping subsampling quantile breakdown point based on blocks of size  $m$ . By definition, in order to get  $m_{MCIV} = \infty$ , we must have  $CIV(m) = \infty$  for all  $m \in \mathcal{M}$ . Given  $m \in \mathcal{M}$ ,  $CIV(m) = \infty$  if and only if the fraction of outliers  $p$  in the sample  $\{X_1, \dots, X_n\}$  satisfies  $p \geq \min\{b_t^{OS}(m - k), b_t^{OS}(m - k + 1), \dots, b_t^{OS}(m + k - 1), b_t^{OS}(m + k)\}$ . This concludes the proof.  $\blacksquare$

**Proof of Corollary 8.** By definition, in order to get  $m_{CM} = \infty$ , we must have  $P[Q_t^{**}(m) = \infty] \geq t$  for all  $m \in \mathcal{M}$ . Given the original sample, Assume that  $q$  nonoverlapping blocks are contaminated with exactly  $\lceil mb \rceil$  outliers for each block, while the remaining  $(r - q)$  are non contaminated (0 outliers), where  $q \in \mathbb{N}$  and  $q \leq r$ . Moreover, assume that the contamination of the contaminated blocks has the structure defined in (28). Let  $X$  be the

random variable which denotes the number of contaminated blocks in the nonoverlapping moving block bootstrap sample. As in (i),  $X \sim \text{BIN}(r, q/r)$ . For the construction of the nonoverlapping moving block bootstrap sample, the selection of  $p \leq r - 1$  contaminated blocks implies the breakdown of  $mp - \lceil mb \rceil + 1$  overlapping subsampling statistics.

$Q_t^{**}(m) = \infty$ , when the proportion of contaminated blocks is larger than  $1 - t$ , i.e.  $\frac{mp - \lceil mb \rceil + 1}{n - m + 1} > 1 - t \Leftrightarrow p > \frac{\lceil (n - m + 1)(1 - t) \rceil + \lceil mb \rceil - 1}{m}$ . This concludes the proof of the second statement. ■

**Proof of Corollary 10.** By definition, in order to get  $m_{HHJ} = \infty$ , we must have  $l_{HHJ} = \infty$ , i.e.,  $MSE(l) = \infty$ , for all  $l \in \{l_{\min} \dots, l_{\max}\}$ . For  $l$  fixed,  $MSE(l) = \infty$  if just a single  $Q_t^*(m, l, i)$ ,  $i = 1, \dots, n - m + 1$  diverges to infinity. This concludes the proof. ■

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$n = 120, b = 0.5$	0.95	0.99
O. Subsampling ( $m = 5$ )	[0.0250; 0.0500]	[0.0250; 0.0250]
O. Subsampling ( $m = 10$ )	[0.0417; 0.0500]	[0.0417; 0.0500]
O. Subsampling ( $m = 15$ )	[0.0667; 0.0667]	[0.0667; 0.0667]
N. Bootstrap ( $m = 5$ )	[0.0250; 0.3750]	[0.0250; 0.3333]
N. Bootstrap ( $m = 10$ )	[0.0417; 0.3333]	[0.0417; 0.2667]
N. Bootstrap ( $m = 15$ )	[0.0667; 0.3250]	[0.0667; 0.2167]
O. Bootstrap ( $m = 5$ )	[0.0250; 0.3750]	[0.0250; 0.3333]
O. Bootstrap ( $m = 10$ )	[0.0417; 0.3333]	[0.0417; 0.2667]
O. Bootstrap ( $m = 15$ )	[0.0667; 0.3250]	[0.0667; 0.2167]

Table 1: **Subsampling and Moving Block Bootstrap Lower and Upper Bounds for the Quantile Breakdown Point.** Breakdown point of the overlapping (O.) subsampling and nonoverlapping (N.) and overlapping (O.) moving block bootstrap quantile. The sample size is  $n = 120$ , the block size  $m = 5, 10, 15$ . We assume a statistic with breakdown point  $b = 0.5$  and confidence level  $t = 0.95, 0.99$ . Lower and upper bounds for quantile breakdown points are computed using Theorem 2 and 3.

$n = 120$	$t = 0.95$	$t = 0.99$
O. Subsampling MCIV	$\leq 0.0500$	$\leq 0.0500$
O. Subsampling CM	$\leq 0.2000$	$\leq 0.2667$
N. Bootstrap HHJ	$\leq 0.1000$	$\leq 0.0667$
O. Bootstrap HHJ	$\leq 0.1000$	$\leq 0.0667$

Table 2: **Breakdown point of Block Size Selection Procedures.** We compute the breakdown point of the minimum confidence index volatility (MCIV), the calibration method (CM), and the data driven method in Hall, Horowitz and Jing (1995) (HHJ) for the nonoverlapping (N.) and overlapping (O.) cases. For (MCIV) and (CM) we use Corollary 6, 8 with  $\mathcal{M} = \{6, 8, 10, 12, 15\}$ . For (HHJ) we use Corollary 10 with  $m = 30$ ,  $l_{min} = 3$ , and  $l_{max} = 10$ . The breakdown point of the statistic is  $b = 0.5$  and the confidence levels are  $t = 0.95, 0.99$ . The sample size is  $n = 120$ .

	(1964 – 2009) (540)	(1964 – 1993) (360)	(1994 – 2009) (180)
Sub. (MCIV)	[0.0247, 0.0520]	[0.0098, 0.0591]	[−0.0290, 0.1995]
Sub. (CM)	[0.0220, 0.0547]	[0.0141, 0.0548]	[−0.0290, 0.1995]
R. F. Sub. (MCIV)	[0.0277, 0.0491]	[0.0177, 0.0509]	[0.0176, 0.1583]
R. F. Sub. (CM)	[0.0268, 0.0501]	[0.0200, 0.0486]	[0.0201, 0.1559]

Table 3: **Stock Returns Predictability.** We report 95% confidence intervals for the parameter  $\beta$  in model (25). We consider the classic subsampling (Sub.) and our robust fast subsampling (R. F. Sub.) with MCIV and CM, for the period 1964-2009 (540 observations) and the subperiods 1964-1993 and 1993-2009 (360 and 180 observations, respectively).

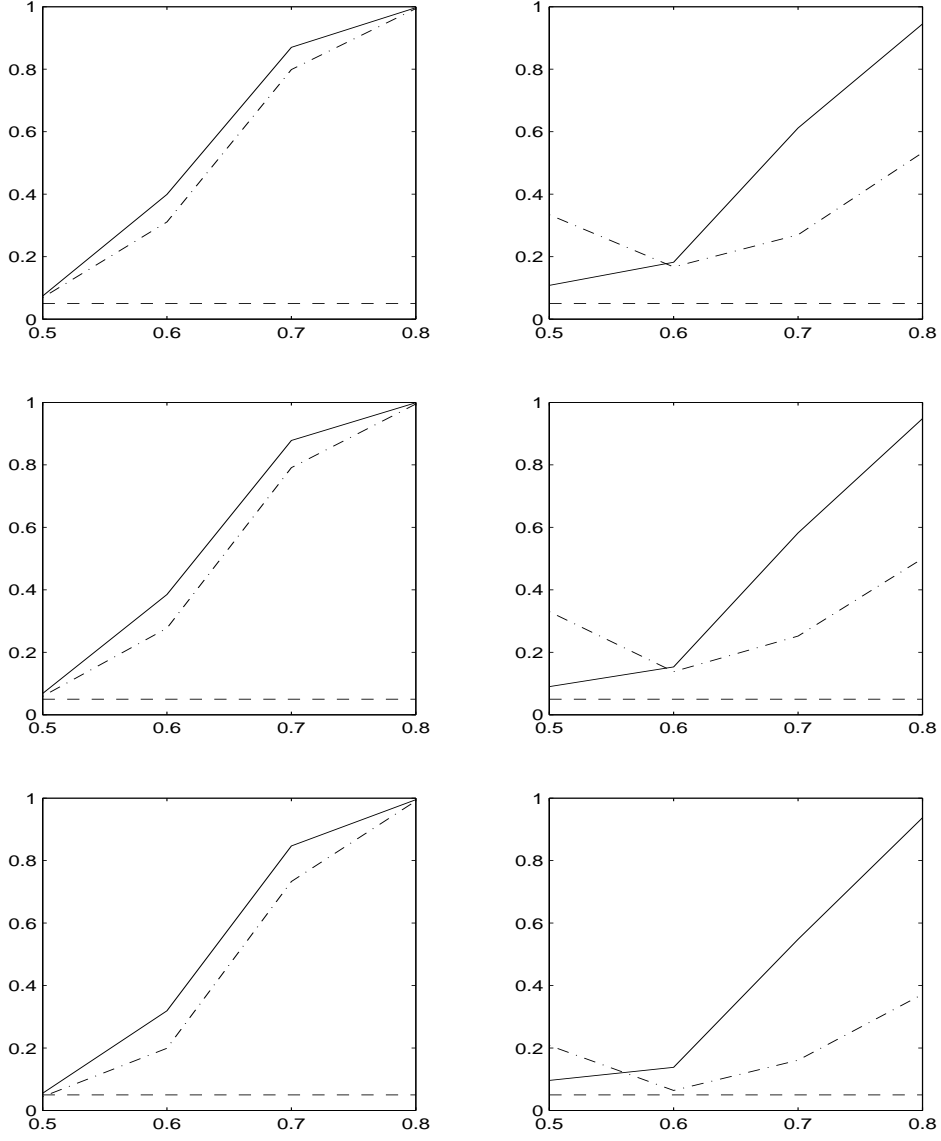


Figure 1: **Power curves in the standard strictly stationary case.** We plot the proportion of rejections of the null hypothesis  $\mathcal{H}_0 : \theta_0 = 0.5$ , when the true parameter value is  $\theta_0 \in \{0.5, 0.6, 0.7, 0.8\}$ . From the top to the bottom, we present the overlapping subsampling with MCIV, the subsampling with CM, and the moving block bootstrap with HHJ. We consider our robust fast approach (straight line) and the classic approach (dash-dotted line). In the left column, we consider a non contaminated sample ( $\eta = 0\%$ ). In the right column, the proportion of outliers is  $\eta = 1.5\%$ .

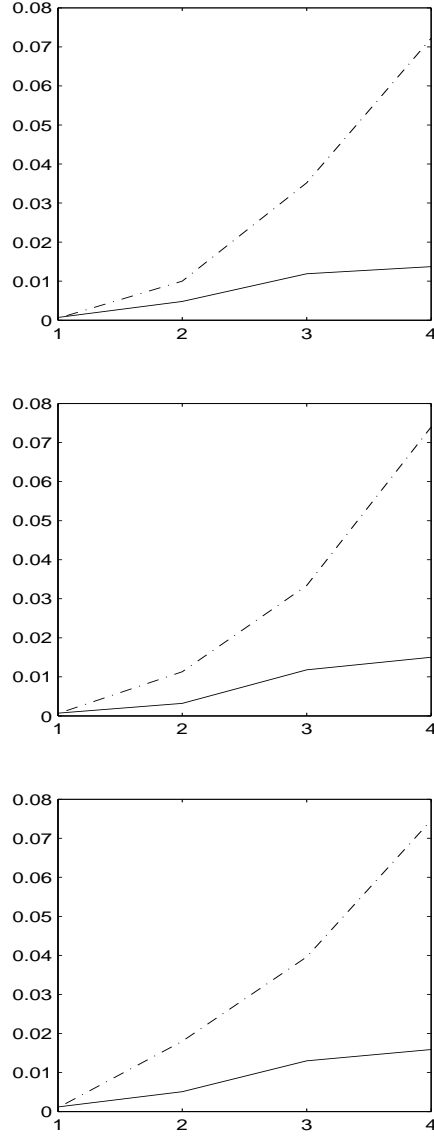


Figure 2: **Sensitivity analysis.** Sensitivity plots of the variation of the empirical  $p$ -value average, for a test of the null hypothesis  $\mathcal{H}_0 : \theta_0 = 0.5$ , with respect to variations of  $X_{\max}$ , in each Monte Carlo sample, within the interval  $\{1, 2, 3, 4\}$ . The random samples are generated under  $\mathcal{H}_0$  and, from the top to the bottom, we present the overlapping subsampling with MCIV, the subsampling with CM, and the moving block bootstrap with HHJ. We consider the robust fast approach (straight line) and the classic nonrobust approach (dash-dotted line).

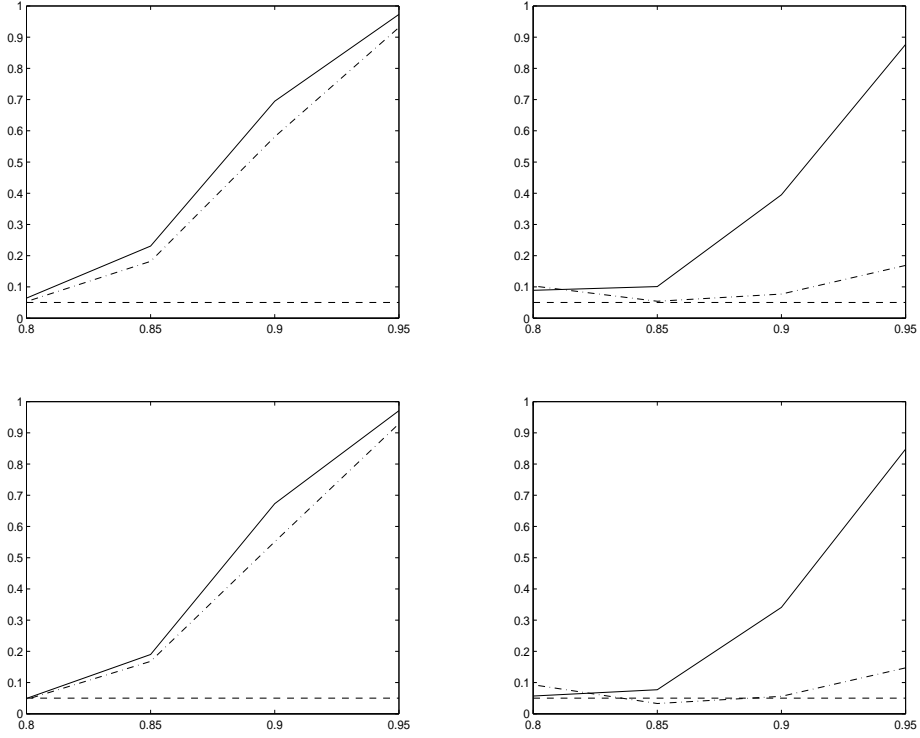


Figure 3: **Power curves in the near-to-unit-root case.** We plot the proportion of rejections of the null hypothesis  $\mathcal{H}_0 : \theta_0 = 0.8$ , when the true parameter value is  $\theta_0 \in \{0.8, 0.85, 0.9, 0.95\}$ . From the top to the bottom, we present the overlapping subsampling with MCIV and CM. We consider our robust fast approach (straight line) and the classic approach (dash-dotted line). In the left column, we consider a noncontaminated sample ( $\eta = 0\%$ ). In the right column, the proportion of outliers is  $\eta = 1.5\%$ .

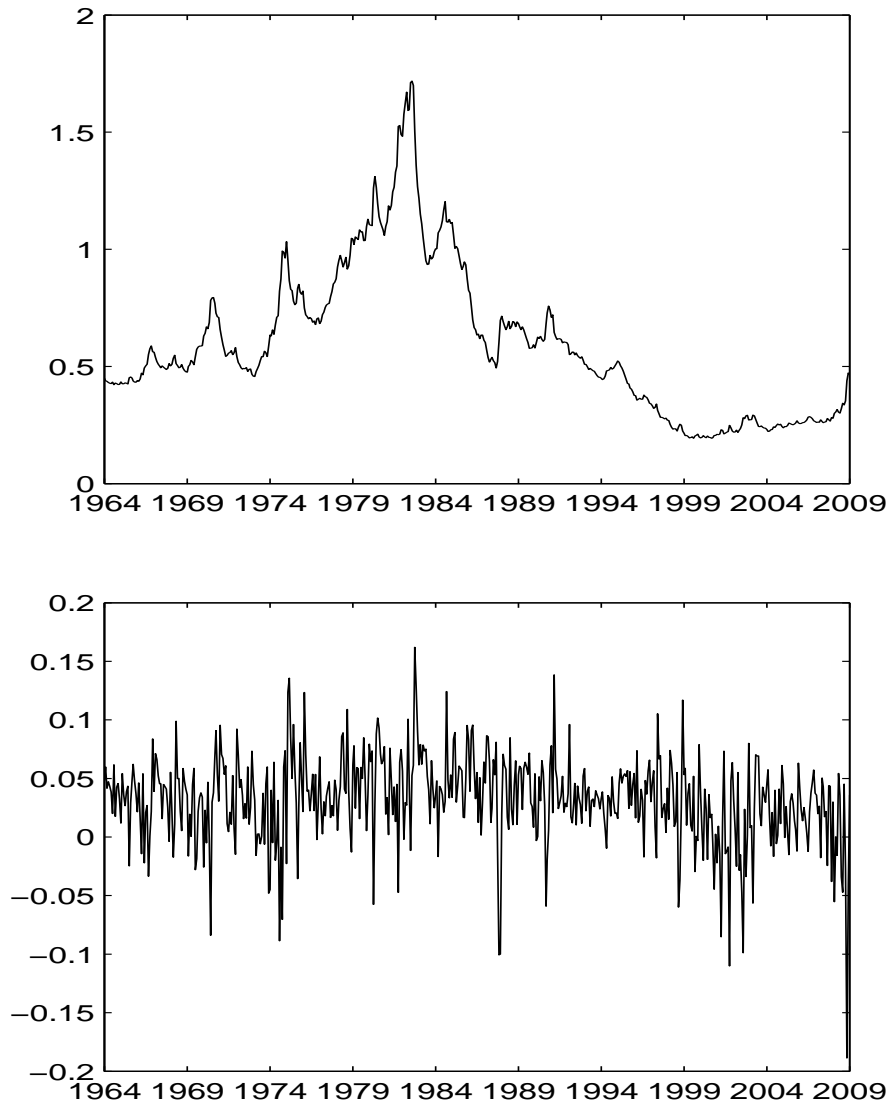


Figure 4: **Dividend yield and log of stock returns.** We plot the dividend yield (top panel) and the log of stock return (bottom panel) of the S&P 500 for the period 1964-2009. We consider monthly S&P 500 index data from Shiller (2000).



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