

## WAVELET MULTIREOLUTION ANALYSIS OF HIGH-FREQUENCY ASIAN FX RATES, SUMMER 1997

### Abstract

FX pricing processes are nonstationary and their frequency characteristics are time-dependent. Most do not conform to geometric Brownian motion, since they exhibit a scaling law with a Hurst exponent between zero and 0.5 and fractal dimensions between 1.5 and 2. This paper uses wavelet multiresolution analysis, with Haar wavelets, to analyze the nonstationarity (time-dependence) and self-similarity (scale-dependence) of intra-day Asian currency spot exchange rates. These are the ask and bid quotes of the currencies of eight Asian countries (Japan, Hong Kong, Indonesia, Malaysia, Philippines, Singapore, Taiwan, Thailand), and of Germany for comparison, for the crisis period May 1, 1998 - August 31, 1997, provided by Telerate (U.S. dollar is the numéraire). Their time-scale dependent spectra, which are localized in time, are observed in wavelet based scalograms. The FX increments can be characterized by the irregularity of their singularities. This degrees of irregularity are measured by homogeneous Hurst exponents. These critical exponents are used to identify the fractal dimension, relative stability and long term dependence of each Asian FX series. The invariance of each identified Hurst exponent is tested by comparing it at varying time and scale (frequency) resolutions. It appears that almost all FX markets show anti-persistent pricing behavior. The anchor currencies of the D-mark and Japanese Yen are ultra-efficient in the sense of being most anti-persistent. The Taiwanese dollar is the most persistent, and thus unpredictable, most likely due to administrative control. FX markets exhibit these non-linear, non-Gaussian dynamic structures, long term dependence, high kurtosis, and high degrees of non-informational (noise) trading, possibly because of frequent capital flows induced by non-synchronized regional business cycles, rapidly changing political risks, unexpected informational shocks to investment opportunities, and, in particular, investment strategies synthesizing interregional claims using cash swaps with different duration horizons.

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## 1 INTRODUCTION

There is insufficient scientific analysis of the crucial financial market phenomena that characterized the Asian Financial Crisis, from a risk management perspective. In particular, the phenomena of financial crises, turbulence, friction and persistence are inadequately described by the conventional ARIMA or geometric Brownian motion models. The main objective of this paper is to present more representative models and analytic procedures, adapted from hydrology, biometrics, and signal processing, to empirically model FX rates.

The measurement of the empirical efficiency of the foreign exchange (FX) markets for risk management purposes dates back to the early 1970s, when the 1944 Breton Woods Agreement of fixed exchange rates was discarded in 1971 and replaced by the current system of flexible exchange rates in 1973 (Cornell and Dietrich, 1978; Friedman and Vandersteel, 1982; McFarland et al., 1982).

In an efficient market, the arrival of new information produces instantaneous price correction, leaving no prospect for price prediction and therefore minimal opportunity for reaping abnormal profits. A market where the best prediction of a price  $\tau$  periods into the future, based on current and past information,  $E_{-\tau} \{x(t + \tau)\}$ , is its current, exact and known, price  $\hat{x}(t) = x(t)$ , is martingale - efficient (Fama, 1970, 1991):<sup>1</sup>

$$E_{-\tau} \{x(t + \tau)\} = \hat{x}(t) \text{ for any } \tau > 0 \quad (1)$$

This implies that the martingale increments

$$\tilde{x}_{-\tau}(t) = x(t + \tau) - \hat{x}(t) \quad (2)$$

have zero mean (per definition):

$$E_{-\tau} \{\tilde{x}_{-\tau}(t)\} = 0 \text{ for any } \tau > 0 \quad (3)$$

and that these increments are independent from each other. They may also, possibly, be identically distributed, or stationary, up to a scale parameter.

Earlier, we investigated the martingale - efficiency of seven Asian FX markets by using non-parametric tests on the historical increments of high frequency, minute-by-minute data (Los, 1999, 2000b) :<sup>2</sup>

$$\tilde{x}_{\tau}(t) = x(t) - \hat{x}(t - \tau) = x(t) - \hat{x}_{\tau}(t) \quad (4)$$

We concluded that:<sup>3</sup>

1. All nine investigated currencies (Hong Kong dollar, Indonesian rupiah, Japanese Yen, Malaysian ringgit, Philippines peso, Singapore dollar, Thai baht, Taiwan dollar, Japanese Yen, and

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<sup>1</sup> We'll introduce some simple notation to connect the various sections of this paper to the literature. It's interesting that some authoritative economists have always doubted the martingale efficiency of financial markets (Grossman and Stiglitz, 1980), on fundamental grounds.

<sup>2</sup> Often analysts investigate only the first-order historical increments  $\tilde{x}_1(t) = x(t) - \hat{x}(t - 1) = x(t) - \hat{x}_1(t)$ , but that is insufficient and leads to rather erroneous conclusions. Investment horizons of different length  $\tau$  are a normal phenomenon in the financial markets.

<sup>3</sup> Los (2000a) did a similar non-parametric analysis of the Asian stock markets.

the benchmark currency of the Deutschemark) exhibit wide sense stationarity in their increments, before the currency turmoil started on the 2nd of July, 1997. When all 1997 FX increments are tested together, only five (Hong Kong dollar, Japanese Yen, Philippines peso, Taiwan dollar and Deutschemark) show wide sense stationarity. The data show a significant discontinuity in the behavior of the increments of four of the currencies (Indonesian rupiah, Malaysian ringgit, Singapore dollar and Thai baht) in the middle of 1997, when Hong Kong was handed over to the People's Republic of China. This suggests that FX models must allow for both stationarity and non-stationarity, in particular for sharp discontinuities.

2. All nine currencies exhibit significant serial dependencies of various lengths  $\tau$  in their increments, in both the whole 1997 data set and in each of the half year data sets, from before and after the mid-year currency break. This suggests that FX models must allow for both serial and long-term (global) dependence.

3. Significant trading windows of up to 20 minutes are identifiable in the FX increments throughout 1997, and more complex persistence behavior than serially dependent Markov processes were identified. This suggests that FX models must allow for more complexity than that of Markov - ARIMA models.<sup>4</sup>

In such FX markets abnormal profits and losses are not only possible. They are, indeed, realized.

This paper examines the long term dependence observed in FX markets, that is not observable by serial, linear dependent (correlation) models of the Markov - ARIMA type. Analysts, who use serial correlation models, often observe that serial FX price residuals are uncorrelated and then, erroneously, conclude that these markets are efficient, because they ignore the more difficult to detect longer - term dependence.

Already in the 1960s, both Mandelbrot and Fama observed that market pricing process are nonstationary. In the early 1970s Brownian motion models became popular, because of the use-

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<sup>4</sup> Any ARIMA model can be translated into a Markov model and vice versa.

fulness of the Nobel Memorial Prize winning Black - Scholes derivative valuation equation. But recently, considerable evidence has been collected that empirical market pricing processes do not conform to arithmetic or geometric Brownian motion, since they exhibit scaling laws with scale exponents that do not conform to Brownian motion, *e.g.*, with Hurst exponents different from 0.5. First, several authors have found that stock market price increments, or rates of return, exhibit empirical Hurst exponents in the range of 0.6 – 0.7. Thus stock market rates of return on such risky assets are persistent.

In contrast, this paper finds that most FX rates exhibit empirical Hurst exponents in the range of 0.3 – 0.5. In particular the anchor currencies of the Deutschmark (now replaced by the Euro) and the Japanese Yen exhibit Hurst exponents of about 0.2 – 0.3. In other words, FX rates are anti-persistent.

From what we have been able to determine from the literature, these are the first empirical measurements of Hurst exponents of FX rates, and the first time for speculative financial markets, of lower than 0.5 are found. These measurements of anti-persistence in the FX markets provide some empirical justification for the use of mean-reverting processes, popular in current dynamic asset pricing theory, to model FX pricing, even though these processes do not capture all global features of long term dependence.

## 1.1 Research Methodology

We use Mallat's (1989) time-scale multiresolution analysis with Haar wavelets (1900) orthonormal filters to analyze such time-scale dependence and self-similarity of the same minute-by-minute indicative quotes of Asian spot currency rates as analyzed in Los (1999, 2000a). Serendipitously, these quotes were systematically collected in Singapore during the whole year 1997 by our students. In this paper, we'll analyze a subset of these high-frequency data in the crucial Asian currency crisis period May 1997- July 1997.

The fractal dimension of each FX market for each of the four months is identified by its

homogeneous Hurst exponent, *i.e.*, its uniform Lipschitz exponent. These critical Hölder exponents measure the persistence and the stability of the fractal nature of the various FX market pricing processes. The uniform Hurst exponents are computed from the multiresolution wavelet coefficients, following the procedure of Kaplan and Kuo (1993).

Since we find significant divergences from independence, FX rate increments do clearly not behave as probabilistic random events. Probabilistic random events are those whose outcomes are determined purely by independent chance. For example, by flipping a fair coin, or, like the statisticians' favorite abstraction: by blindfoldedly taking colored balls from a small urn. If we assume that all balls are exactly the same except for color, then each ball is equally likely to be chosen, so the selection process is probabilistically random.<sup>5</sup>

However, we find that current FX increments have already been impacted by a number of previous increments, although not with precise periodicity. The precise number of these impacts depends on the size of the temporal windows used, when we identify this divergence from independence. Similarly, the current FX increments will also impact future FX increments in a non-periodic, but cyclic fashion.

Thus, the conventional statistical abstraction of probabilistic randomness cannot function as a null hypothesis. The observation of particular dependent FX rates increments conditions and limits the observable distribution, which may be "random" only within the finite constraints of a new frequency histogram. The observable distribution can thus only be conditionally and not unconditionally "random."<sup>6</sup> Therefore, we opt in this paper for an approach to the measurement of the randomness of FX data borrowed from scientific signal processing, which relies on the concept of, precisely defined, measured irregularity. This scientific approach does not rely on "significance" testing based on assumed probability or on the introduction of extraneous "degrees

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<sup>5</sup> This deplorable connection between probability and randomness - two very different concepts - was introduced by the 16th century physician Girolamo Cardano (1500 - 1571) in his *Liber de Ludo Aleae* (*Book on the Games of Chance*), posthumously published in 1663 (Cf. Bernstein, 1996, p. 47 - 50 and 53 - 55).

<sup>6</sup> As GARCH models recognize to a certain degree. But the processes produced by GARCH models do not have measurement characteristics that match that of empirically observed FX increments (Peters, 1994).

of confidence.”

## 1.2 Empirical Characteristics of FX Rates

Four empirical characteristics of the FX rates justify this scientific irregularity measurement approach, as this paper will demonstrate:

1. FX rates are conspicuously discontinuous, *i.e.*, singular at almost every point, because the supply and demand curves of currencies move in often small, but sometimes large, discrete steps (”ticks”), in instantaneous response to news events. Therefore, continuous time dynamic Itô (e.g., arithmetic or geometric Brownian motion) pricing models appear to be scientifically inappropriate. In addition, the Poisson-type jump processes proposed by Jorion (1988) are too restrictive, since it’s impossible to determine with any degree of certainty the Poisson rates from finite empirical data sets.

2. The distributions of FX rates are strictly nonstationary. However, they clearly adhere to stable scaling, or (Pareto-Lévi type) power laws and are thus stationary at several scales.

3. FX rates show non-periodic cyclicity, *i.e.*, intermittent periods of condensation, succeeded by periods of rarefaction.<sup>7</sup>

4. FX rates are finite, in two senses: even high-frequency data sets have a finite number of observations and exhibit finite amplitudes.

Such high-frequency, singular price data are similar to physiological measurement data, such as heart records, electromagnetic fluctuations in galactic radiation noise, textures in images of natural terrain, variations of electric grid or traffic flows, etc. (Mandelbrot, 1999). But not all such series of singularities are alike. Knowing the degree of irregularity of such singularities is important in analyzing their risk properties. For the purpose of risk management, knowing the distribution of the degrees of irregularity of financial time series is crucial for a correct analysis of the non-periodic, but cyclic financial risk.

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<sup>7</sup> This is a major area for our current research, and clearly inspired by the early work by Mandelbrot in the 1960s.

Therefore, we introduce a proper formal definition of measured irregularity, or randomness, as measured by the Lipschitz regularity exponent  $\alpha_L$ . Unfortunately, direct pointwise measurements of Lipschitz irregularity exponents, which measure the degree of irregularity of singularities, are not possible, because of the finite numerical resolution. After discretization, each data set corresponds to a time interval where the time series has a very large ("infinite") number of singularities, which may all be different. Thus, in general, a singularity distribution, or singularity spectrum, should be computed from global measurements, which take advantage of multifractal self-similarities. However that's the subject of another paper. The objective of this paper is much more limited and pragmatic.

Mandelbrot and Van Ness (1968) find that fractional Brownian motion (FBM) provides a convenient model for such self-similar time series. Hosking (181) incorporates the FBM into his ARFIMA model. FBMs are statistically self-similar i.i.d. processes, which exhibit long-term dependence. Despite their nonstationarity, their power spectrum is defined. It shows power decay: FBMs exhibit  $\frac{1}{\omega}$ -type spectral behavior over wide ranges of radian frequencies  $\omega$ . Realizations of FBMs are almost everywhere singular, with the same uniform  $\alpha_L$ -Lipschitz regularity at all points.

Because of the singularity structure and finiteness of the FX data, Fourier analysis, even of the Gabor-windowed kind, cannot be used to detect the precise timing of the non-stationarities and self-similarities. Instead, wavelet multiresolution analysis appears to be the relevant analysis tool. Wavelet multiresolution analysis can be applied to data sets of any finite length of discontinuous and singular observations.

### **1.3 Organization of the Paper**

This paper is organized as follows. In Section 2 we review the relevant literature not yet reviewed in Los (1999, 2000a and b), which we organize around three relevant questions for FX market research:

- (1) What distributions of high frequency FX rate increments are produced by the FX markets?

Are they linear or nonlinear, Gaussian or non-Gaussian?

(2) What is the degree of stationarity of these FX increments?

(2) What is the degree of their temporal, long term, dependence?

. This is followed by a tabular and graphical presentation of the data characteristics of the high frequency FX rate increments in Section 3 , as well as a detailed and rather pedagogical discussion of the proposed analytic methodology in Section 4. Subsequently, analyzing the data with the new tools, we obtain three major results in Section 5:

1. The non-Japanese Asian FX markets show evidence of a non-linear, non-Gaussian FX increments with occasional persistence, because of infrequent trading and prevalent illiquidity.

2. Some FX increments are stationary, others are non-stationary.

3. Virtually all FX increments under investigation are anti-persistent, except that of the Taiwan dollar. The fast-trading Japanese Yen and Deutschemark anchor currency markets are ultra-efficient, in the sense that they are invariant anti-persistent.

These results are summarized in Table 3 in Section 5 and discussed and interpreted in great detail FX market . This is followed by conclusions and suggestions for future research.

## **2 LITERATURE REVIEW**

Numerous anomalies reported in the finance literature contradict the random walk model of Bachelier (1900). The geometric Brownian motion, assumes stationarity and independence of the serial increments of rates, and allows only positive prices. But such Brownian motion is a poor description of the price behavior of financial instruments, since empirical FX increment series, display skewness and high kurtosis, features not explained by the Wiener processes driving Brownian motion. Goodhart and Figlioli (1991) note that increasing the sample size and the frequency of observations tends to reduce the two higher moments. But Boothe and Glassman (1987) reject any single stable distribution. Most researchers now accept that FX rates have mixed distributions that may vary with time (Giovanni and Jorin, 1989) and that exhibit some form of scaling law

(Müller, et al.,1990).

Thus FX rate increments are not Wiener processes (Feinstone, 1987), nor are FX rates Markov processes (Müller et al. 1990). The FX rates substantial heteroskedasticity and are non-stationary in, at least, the first four moments (Hsieh, 1989). Such non-linear processes could be caused by time-varying variance or by variable memory processes (Attansio, 1991). Therefore, Engle et al. (1990), Baillie and Bollerslev (1989a & b, 1991) and Diebold and Nerlove (1989) propose that FX series are conditional expectation (GARCH) processes. However, this paper shows clear evidence of mixed serial and long term dependent processes and conditional expectation processes do not foot the empirical observations (Peters, 1994).

Ashley et. al. (1989) and Meese and Rose (1991) point out that financial time series reveal a non-linear structure. Hamilton (1989) attributes this to the non-linear dynamic structures generating the data, while Antoniou and Holmes (1997) attribute it more specifically to thin trading.<sup>8</sup> This paper corroborates the nonlinearity, non-Gaussianity and the thin trading of FX rate generating processes in Asian emerging markets.

Anderson and Bollerslev (1997), as well as Müller et al. (1990, 1998), clearly demonstrate the presence of long term dependence and heavy tailed distributions in high frequency financial data. The presence of long term dependence can be quantified by the Hurst exponent for a limited range of values. Müller et al. (1990) prove that scaling of the variance of the increments by a Hurst exponent of 0.5 (= square root scaling) leads to mispricing. This seriously questions the validity of the currently popular dynamic pricing models (*e.g.*, in Duffie, 1996), which linearly scale the risk (measured by the variance  $\sigma_{\xi}^2$ ) from other time periods. It is important to note that fractality in FX rates corresponds primarily to scaling laws defined in the time domain, as shown by Müller et al. (1990). This paper shows that the Hurst exponent is below 0.5 for FX rates.

This latest scientific evidence leads to the following specific research question, first raised by Mandelbrot (1963, 1966, 1971) and, more recently, by Peters (1994): is the Fractal Market

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<sup>8</sup> These two arguments may amount to the same!

Hypothesis (FMH), based on the Fractal Brownian Motion (FBM) a viable alternative to the Efficient Market Hypothesis (EMH), based on Fama's martingale theory, in particular, the geometric Brownian motion, which is now clearly rejected by empirical data? Cheung (1993) and Batten et al. (1999) show additional and very convincing evidence of fractal distributions in FX markets. This paper uses a more sophisticated methodology and comes to very similar, but more detailed conclusions for Asian FX markets that are generally considered inefficient.

In the past decade, there has emerged a strong interest in computing wavelet multiresolution of financial data. Wavelet transforms can pertinently capture the time-varying spectral decomposition of non-stationary signals. They can also identify the spatial inhomogeneities (the trend and the structural shifts) which are important common phenomena in financial pricing series (Jorion, 1988) and they can deal with singularities and discontinuities. Ramsey and Lampart (1998a & b) have computed wavelet multiresolutions of various well-known macroeconomic data series. Ramsey et al. (1995) and Ramsey and Zhang (1997) use wavelets and scalograms to demonstrate the robust evidence of self-similarity financial time series.

### **3 DATA CHARACTERISTICS**

Indicative minute-by-minute FX quotes Deutschemark (DEM) and of eight Asian currencies, namely Hong Kong dollar (HKD), Indonesian rupiah (IDR), Japanese Yen (JPY), Malaysian ringgit (MYR), Philippine peso (PHP), Singapore dollar (SGD), Thai baht (THB), and Taiwanese dollar (TWD), versus the American dollar (USD) numéraire, were collected during 1997 from Dow Jones Telerate, in the Simulated Trading Room of Nanyang Technological University, from January 1 through December 31, 1997, with the exception of five days in October, 1997.<sup>9</sup> However, in this paper we use only four focused months of data from this unique data set. There are  $30 \times 24 \times 60 = 43,200$  minute-by-minute observations per 30-day month.

Unfortunately, only indicative quotes could be collected. Foster et al. (1993) and Evans (1997)

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<sup>9</sup> Due to a technical disruption in the NTU on-campus Simulated Trading Room, there was an unfortunate lapse of data availability from 22 to 31 October 1997, missing 5 days of trading.

claim that there is little or no evidence of a relationship between volume (= size of the transactions) and volatility (Demos and Goodhart, 1992), in contrast to Bollerslev and Domowitz (1993) and Bollerslev and Melvin (1994), while maintaining that the density of the transactions is a better indicator of their volatility. That claim prefaces our conclusion regarding the importance of trading frequency. Since we had no volume data, we could not further investigate this relationship between trading frequency and volume, which may be important for forecasting (Engle and Russell, 1997).

These high frequency FX rates, coming at irregular time intervals, were made regular by retaining the first FX quote within each minute interval and by repeating the same value for the minutes showing no transaction, thereby creating zero increments for those time intervals. Thus the FX rates remain unchanged from the last transaction until there is a new nonzero increment. Such time regularization does not introduce additional volatility, since it does not change the increments, although it might increase the number of transactions and therefore alter somewhat the estimates of the third and fourth moments, a problem called aliasing by signal processing engineers. Aware of these possible small distortions, time regularization of data, by turning them into a step function, has the advantage that one can examine the data more clearly using wavelets (Ogden, 1997).<sup>10</sup>

This paper uses the mid of the bid-ask spread of these FX quotes, considering that these are vary mostly parallel in the FX markets (Bessembinder, 1994). Initially tick rates were used, but they were found to be too voluminous to work with within the time constraints of this study and we settled for a standardized one minute time interval  $\Delta t$ . There are still very few studies focusing on Asia-Pacific currencies. However the recent financial turmoil in Asia clearly calls for more and better analysis of the microstructures and trading patterns in Asian FX rates. The German FX rates, deemed a priori to be more efficient, are examined for comparison.

According to the FX surveys of the Bank of International Settlement (BIS) of 1995 and 1998,

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<sup>10</sup> In the next paper, we'll analyze tick-by-tick transaction data directly, when we compute singularity spectra directly.

the DEM and JPY are heavily traded currency pairs, or anchor currencies. In addition, only Singapore FX activity and Hong Kong were significant. Even in these markets it was mostly the three anchor currencies (USD, DEM, JPY) that were traded. In general, the BIS surveys clearly show that most Asian currencies lack liquidity.

However, there some changes in trading occurred after the Asian Financial Crisis and the introduction of the Euro in January 1999. For most Asian currencies, the trading fell drastically after the midsummer 1997 currency break, due to capital restrictions, *e.g.*, in Malaysia in September 1997, and drying up of business due to negative market sentiments. It is noteworthy to point out that the trading for PHP increased after the currency break. The PHP was the most thinly traded with barely more than 100 transactions per month before the break. Thus any findings pertaining to PHP should be treated with great caution.

Of course, the DEM shows less trading activity from the last quarter of 1998 on, as it is being replaced by the Euro. A graph on page 81 of *The Economist* of September 16th, 2000, shows that since 1985 until the present the unofficial, and since January 1, 1999, the official Euro has traded virtually parallel with the Deutschemark, with a very gradual narrowing of the value gap between the two.

### **3.1 Distributional Statistics**

FX rates have two dimensions: a frequency, or distributional, dimension and a time dimension. These two dimensions are presented consecutively. From Table 1, it is evident that the DEM and JPY, during the sample period, were generally less volatile than other Asian currencies, regardless of the onset of the Asian Financial Crisis. Generally the other Asian currencies, especially the THB and IDR, experienced higher volatility after the floating of the THB on July 2, 1997.

[TABLE 1 about here]

All increments reveal high kurtosis and hence they are not Gaussian processes, even after detrending. Table 1 shows that DEM, JPY and HKD are closer to Gaussianity than other Asian

currencies. As expected Gaussianity varied from month to month. Nevertheless a zero mean Gaussian process may still be a fair approximation for most currencies, when ignoring long term dependencies, with the exception of IDR in May 1997. The reason for the extraordinary kurtosis of IDR for the month of May 1997 is that in that month the Indonesian rupiah hardly traded at all. The IDR increments in May were zero most of the time, interspersed with a few very large increments.

We performed also higher-order spectral analysis tests for both Gaussianity and linearity using the higher-order cumulants of the series. If a time series is Gaussian, its third and subsequent cumulants will be zero. In the Gaussianity tests, the null hypothesis is that the data exhibits a zero bispectrum. Though the estimates will not be exactly zero, the estimated quantities are statistically tested for significantly different from zero. For the linearity test, the inter-quartile range of the estimated bicoherence ( $B$ ) is computed. The absolute value of the bicoherence is a constant if the data is linear, non-Gaussian and one determines whether the observed variation of  $B$  is statistically significant.

[TABLE 2 about here]

From the tests in Table 2, we find that most FX increments, with the exception of the DEM, are non-Gaussian although about half are deemed linear. However, the linearity varies among the currencies and according to time: July (after the break) shows more non-linearity than June (before the break). THB exhibits clearly non-Gaussianity and non-linearity. IDR, MYR, PHP, TWD show an abrupt change in the tests after July 2, 1997: they are non-linear. Gaussianity is rejected, in most cases, due to the exceptionally high kurtosis.

Fig. 1. shows the histogram of JPY June 1997 and a normal distribution is fitted. It clearly visualizes that Gaussianity is violated due to high kurtosis. There are more small increments (noise trading), more large increments (occasional outlying FX rate changes), and less moderately sized increments than the normal distribution suggests, even in a very efficient anchor currency

market as the Japanese Yen. The histograms of the Deutschemark is almost identical.<sup>11</sup>

[FIGURE 1 about here]

### 3.2 Power Spectra and Spectrograms

Spectral representation decomposes a time series into a sum of sinusoidal components with Fourier coefficients. A plot of a power spectrum gives the squared Fourier coefficients versus trading frequency  $\omega$ , where is the contribution to the total power from the term in the Fourier series with trading frequency cycles per minute.

Initially, we computed the power spectral densities of the original FX series for June and July 1997. All FX rates very clearly show exponential decay over their frequency ranges. As the power spectra for the JPY in Fig. 2 show, most power of the original FX rates resides in the low frequencies (*e.g.*, FX rates have unit roots, in addition to fractional roots) and decays exponentially. In contrast, the power of the FX increments increases with the frequency: the highest frequencies have most power.<sup>12</sup> But such power spectra don't tell the whole story, since they assume stationarity in the FX series and ignore time dependence.

[FIGURE 2 about here]

Because of the combined frequency and time dimensions of FX rates, and the potential for non-stationarities, we generate spectrograms for June and July 1997, using Gabor's windowed Fourier Transforms, with a time-localization Hanning window. By moving this window along the time dimension of the FX rate increments, smooth variations of the spectrum as a function of time can be visualized.

We also compute a spectrogram of white noise with mean zero and a constant variance similar to that of DEM June 1997 (*i.e.*, with the same size of variance of  $1.6e-05$ ) to provide a comparison

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<sup>11</sup> The histograms of all investigated currencies are available upon request.

<sup>12</sup> All power spectral density plots are available upon request. These plots show the frequencies measured along the horizontal axis from low at the left to high at the right, and standardized from 0 to 1. The magnitude is measured along the vertical axis.

standard (Fig. 3).<sup>13</sup> From the spectrogram of the FX increments in the left panel of Fig. 3, it is evident that the FX increments are clearly not white noise. Most power of the FX increments is present mostly in the higher frequencies at the top of the spectrograms and it is not evenly distributed over time. For the white noise in the right panel of Fig. 3, power is constant over all frequencies and all times.

[FIGURE 3 about here]

From the spectrograms, it is evident that the FX increments, and therefore the various FX pricing processes, are very different from one other.<sup>14</sup> However the spectrograms of DEM and JPY do not differ much from each other in June 1997. In July the JPY displays less power in the lower frequencies as compared to JPY in June 1997. However, the DEM exhibits more power in the lower frequencies for July 1997. As for the HKD, there is more power in the later half of June 1997. Power is concentrated in the higher frequencies for HKD July 1997. The SGD does not differ much in the observed periods. In July 1997, the IDR, MYR, PHP and TWD reveal the change in frequencies, which indicate the non-stationarity of the increments. The THB reveals a change in trading frequencies from June 1997 to July 1997.

However, we could only determine that the FX increments are not white noise. Whether they represent anti-persistent or persistent noise cannot be determined from spectrograms. For an unambiguous answer to that question we needed to apply wavelet multiresolution analysis, which is a time-scale analysis. Time-scale analysis translates frequency into scales ( $\text{scale} = 1/\omega$ ).

## 4 ANALYTIC METHODOLOGY

Since most financial economists are unfamiliar with the mathematical apparatus of Lipschitz irregularity and Hurst exponents, fractal Brownian motion (FBM), and wavelet multiresolution analysis (MRA), this section provides a concise pedagogical review of these analytical concepts.

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<sup>13</sup> The color coding is as follows: low power is coded by blue; high power is coded by red; and intermediate power is yellow. The frequencies are standardized between 0 and 1, 0 being low frequencies and close to 1 being high frequencies..

<sup>14</sup> The spectrograms of all FX increments are available upon request.

## 4.1 Measurement of Lipschitz Irregularity

Our FX increments are locally very irregular: they are continuous time series, but their first derivatives exists almost nowhere, *i.e.*, their increments are singularities almost everywhere. But are they Wiener increments? We need a precise measurement apparatus to empirically measure their degree of irregularity or "randomness." A time series is called *regular* if it can be locally approximated by a polynomial. If not, it is called *irregular*.

Of course, there are degrees of irregularity, from highly irregular to highly regular. These degrees of irregularity can be measured by the Lipschitz regularity exponent  $\alpha_L$ . If the time series  $x(t)$  has a singularity at time  $\tau$ , *i.e.*, if it is not differentiable at  $\tau$ , the Lipschitz regularity exponent  $\alpha_L$  characterizes this singular behavior at time  $\tau$ . When we measure the Lipschitz  $\alpha_L$  of a singularity, we can still assess how irregular, or random, such a singularity is. In other words, we no longer have to assume the degree of randomness of a time series, as done by conventional probability based statistics. We measure its randomness by computing its Lipschitz  $\alpha_L$ !

The Lipschitz regularity exponent  $\alpha_L$  is based on the approximation error of the familiar Taylor expansion formula, which relates the differentiability of the continuous time series  $x(t)$  to a local polynomial approximation.

**Definition 2** Suppose that  $x(t)$  is  $d$  times differentiable in the bounded interval  $[\tau - \epsilon, \tau + \epsilon]$  for small  $\epsilon > 0$ . Then we can expand  $x(t)$  as follows:

$$\begin{aligned} x(t) &= \left[ \sum_{k=0}^{d-1} \frac{x^{(k)}(\tau)}{k!} (t - \tau)^k \right] + \tilde{x}_\tau(t) \\ &= \hat{x}_\tau(t) + \tilde{x}_\tau(t) \end{aligned} \tag{5}$$

where  $x^{(k)}(t)$  is the  $k$ -th derivative of the data series  $x(t)$ . The  $\hat{x}_\tau(t)$  is the exact Taylor polynomial expansion, or model, of  $x(t)$  at time  $\tau$  and  $\tilde{x}_\tau(t) = x(t) - \hat{x}_\tau(t)$  is the **approximation error**, or the inexact, irregular, random, unpredictable part of the data.<sup>15</sup> The approximation error  $\tilde{x}_\tau(t)$  satisfies the inequality

$$\tilde{x}_\tau(t) \leq \sup_{u \in [\tau - \epsilon, \tau + \epsilon]} \left| x^{(d)}(u) \right| \frac{|t - \tau|^d}{d!} \text{ for all } t \in [\tau - \epsilon, \tau + \epsilon] \tag{6}$$

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<sup>15</sup> Statisticians often call the approximation error  $\epsilon_\tau(t)$  the residual. It is clear that the character of this residual depends on the number of differentiation terms included in the linear Taylor expansion. Therefore, one cannot ascribe inherent characteristics like "normally distributiveness" to this residual, since such characteristics are not *sui generis*. Still, this is what statisticians conventionally do.

The  $d^{\text{th}}$ -order differentiability of  $x(t)$  in the neighborhood of  $\tau$  yields an upper bound on the approximation error  $\tilde{x}_\tau(t)$ , when  $t$  tends to  $\tau$ , *i.e.*, when the time interval  $\Delta_\tau t = t - \tau$  becomes smaller. The Lipschitz regularity refines this upper bound with the non-integer difference order  $d$ .

**Definition 3** A time series  $x(t)$  is **pointwise  $\alpha_L$ -Lipschitz regular**, with regularity exponent  $\alpha_L \geq 0$  at point  $\tau$ , if there exists a  $K > 0$ , and a polynomial  $\hat{x}_\tau$  of degree  $d = \lfloor \alpha_L \rfloor$  such that for all  $t \in \mathbb{R}$ , the absolute value of the error

$$|\tilde{x}_\tau(t)| = |x(t) - \hat{x}_\tau(t)| \leq K |t - \tau|^d = K |t - \tau|^{\alpha_L} \quad (7)$$

or, equivalently, if

$$|x(t) - \hat{x}_\tau(t)|^{-\alpha_L} \leq K |t - \tau| \quad (8)$$

**Definition 4** A time series  $x(t)$  is **uniformly  $\alpha_L$ -Lipschitz regular** over the interval  $[a, b]$  if it is pointwise Lipschitz  $\alpha_L$  for all  $\tau \in [a, b]$ , with a constant  $K$  that is independent of  $\tau$ .

**Definition 5** The **Lipschitz regularity exponent** of  $x(t)$  at point  $\tau$  or over the interval  $[a, b]$  is the supremum of  $\alpha_L$  such that  $x(t)$  is  $\alpha_L$ -Lipschitz (pointwise or uniformly).

This is a technical definition of (ir-)regularity, which requires additional explication. At each time point  $\tau$ , the polynomial  $\hat{x}_\tau(t)$  is uniquely defined. If  $x(t)$  is  $d = \lfloor \alpha_L \rfloor$  times continuously differentiable in the neighborhood of  $\tau$ , then  $\hat{x}_\tau(t)$  is the linear Taylor expansion of  $x(t)$  at  $\tau$ . Thus when  $\alpha_L$  is an integer, the regularity at point  $\tau$  is defined as usual, with  $\alpha_L$  indicating the order of differentiability of  $x(t)$ . When  $\alpha_L$  is a fraction, let  $d$  be an integer such that  $d < \alpha_L < d + 1$ , then  $x(t)$  has an  $\alpha_L$ -Lipschitz regularity at  $\tau$ , if its derivative  $x(t)^{(d)}$  of order  $d$  resembles  $|t - \tau|^{\alpha_L - d}$  locally around point  $\tau$ .

The degree of regularity of  $x(t)$  in a domain is that of its least regular point. The greater  $\alpha_L$ , the more regular the time series  $x(t)$ . The smaller  $\alpha_L$ , the more irregular the time series  $x(t)$ . There exist multifractal time series  $x(t)$  with non-isolated singularities, where  $x(t)$  has a different Lipschitz  $\alpha_L$  at each point  $\tau$ . In contrast, uniform Lipschitz  $\alpha_L$  exponents provide a more global measurement of irregularity, which applies to a whole interval.<sup>16</sup> If  $x(t)$  is uniformly Lipschitz  $\alpha_L > d$ , where  $d$  is an integer, then one can verify that  $x(t)$  is  $d$  times continuously differentiable in this neighborhood.

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<sup>16</sup> At this moment it is an open empirical research question if FX markets are pointwise singular or uniformly singular. In this paper we maintain that FX series are uniform singular, although our empirical evidence appears to suggest that they are pointwise singular.

What values of the Lipschitz exponent  $\alpha_L$  should one expect for the singularities in the series of FX increments? If the Lipschitz exponent is a fraction,  $0 \leq \alpha_L < 1$ , then  $\hat{x}_\tau(t) = x(\tau)$  and the Lipschitz condition simplifies to:

$$|x(t) - x(\tau)| \leq K |t - \tau|^{\alpha_L} \text{ for all } t \in \mathbb{R} \quad (9)$$

A time series  $x(t)$  that is bounded, but discontinuous at time  $\tau$ , has Lipschitz exponent  $\alpha_L = 0$  at  $\tau$ . If the Lipschitz irregularity is  $0 < \alpha_L < 1$  at  $\tau$ , then  $x(t)$  is continuous, but not differentiable at  $\tau$  and  $\alpha_L$  characterizes this particular singularity type. Since for this paper we have regularized the FX series under examination, *i.e.*, we have made the FX series continuous, we want to compute the degree of their irregularity by computing the uniform Lipschitz exponent  $\alpha_L$ , which we expect to be a fraction.

## 4.2 Fractional Brownian Motion

But what degree of polynomial should we use as approximation model for the FX rates? The self-similarity, or scaling property of fractional Brownian motion (FBM) model appears to fit the data characteristics described in Sections 1 and 3, although more general models are possible. Many researchers have found unit root phenomena in FX rates, thus a first-order differentiation is corroborated and we can focus on the fractional irregular FX increments.

As Peters (1994) suggests, optimal consumption, savings and portfolio investment decisions may be extremely sensitive to investment horizons  $\tau$ , when the investment returns are long-term dependent. Problems may arise with the risk-neutral valuation of primary and derivative securities (such as options and futures) based on Fama's (1970, 1991) martingale probabilities. As we noted in Section 1. the continuous-time stochastic processes most commonly employed in such valuation models, *e.g.*, geometric Brownian motions, are inconsistent with long-term dependencies. But why?

With the empirical characteristics of FX rates described in Section 2, traditional tests are no longer valid, since the usual forms of statistical inference do not apply to time series exhibiting

long term persistence (Lo & MacKinlay, 1999). Mandelbrot (1971) was the first to consider the implications of such long term dependence in asset returns for the martingale definition of financial market efficiency. His seminal research has recently acquired greater attention, because of the apparent non-periodic cyclicity of financial crises, which occur with much greater frequency and impact than normally expected. The question arises: when such periods of financial crises and financial turbulence, which intersperse periods of relative trading tranquility, are unpredictable, can at least their volatility (risk) be modeled and thus valued? In other words, can the long term dependence of the financial market risk be modeled for valuation purposes?

The FBM is a non-stationary process with an infinite time span of temporal dependence. The FBM was originally proposed by Mandelbrot and Van Ness (1968). Hosking (1981) encompassed the FBM by his Autoregressive Fractionally Integrated Moving Average, or ARFIMA( $p, d, q$ ) model, with  $d =$  a real fraction, where long-term, low-frequency, long memory processes are superimposed on short-term or high frequency effects. These fractionally differenced, stochastic processes are not strong-mixing. They are non-stationary, but have a power spectrum with a power decay. Their autocorrelation functions (ACFs) decay at much slower rates than those of the serial, weakly dependent processes.

**Definition 6** A *Fractionally Differenced Time Series* is defined by

$$(1 - L)^d X(t) = \varepsilon(t) \tag{10}$$

where  $L$  is the lag operator and  $d > 0$  is a real fraction  $\in \mathbb{R}$ .

As Lo & MacKinlay (1999) show, we can view this process as an infinite-order moving-average (MA) process, since

$$X(t) = (1 - L)^{-d} \varepsilon(t) \tag{11}$$

$$= \sum_{\tau=0}^{\infty} b(\tau) L^\tau \varepsilon(t), \text{ with } \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{12}$$

*i.e.*, a weighted summation of white noise  $\varepsilon(t)$ , where the MA coefficients  $b(\tau)$  can be expressed

in terms of the gamma function<sup>17</sup>

$$\begin{aligned} b(\tau) &= (-1)^\tau \begin{pmatrix} -d \\ \tau \end{pmatrix} \\ &= \frac{\Gamma(\tau + d)}{\Gamma(d)\Gamma(\tau + 1)} \end{aligned} \quad (13)$$

Thus, we have arrived at a useful definition of a long-term dependent random or irregular process.

**Definition 7** *A long-term dependent random or irregular process is a process with an autocovariance function (ACF)  $\gamma(\tau)$ , such that*

$$\begin{aligned} \gamma(\tau) &= E \{x(t)x(t - \tau)\} \\ &= \int_{-\infty}^{\infty} x(t)x(t - \tau) \\ &= \left\{ \begin{array}{l} \tau^\lambda L(\tau) \text{ for } \lambda \in [-1, 0) \\ -\tau^\lambda L(\tau) \text{ for } \lambda \in (-2, -1] \end{array} \right\} \end{aligned} \quad (14)$$

as the time interval lengthens,  $\tau \rightarrow \infty$ , where  $L(\tau)$  is any slowly varying function at infinity, e.g., a constant.

The ACF of Hosking's (1981) fractionally-differenced time series, when  $\varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2)$ , is given by:

$$\begin{aligned} \gamma(\tau) &= \frac{\sigma_\varepsilon^2 (-1)^\tau (-2d)!}{(\tau - d)! (-\tau - d)!} \\ &= \frac{\sigma_\varepsilon^2 \Gamma(1 - 2d) \Gamma(\tau + d)}{\Gamma(d) \Gamma(1 - d) \Gamma(\tau + 1 - d)} \\ &\sim \sigma_\varepsilon^2 \tau^{-2d-1} \text{ as } \tau \rightarrow \infty \end{aligned} \quad (15)$$

where  $d \in (-\frac{1}{2}, \frac{1}{2})$ . Thus, the dependence exponent  $\lambda = 2d - 1$  and the slowly varying function is the constant white noise variance  $\sigma_\varepsilon^2$ .

This ACF is slowly decaying. When  $d \downarrow -\frac{1}{2}$ , the white noise  $\varepsilon(t)$  is fractionally differentiated, and the ACF decays faster than hyperbolically,  $\gamma(\tau) \rightarrow \sigma_\varepsilon^2 \tau^{-2}$ , and the series of increments

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<sup>17</sup> The gamma function has the property

$$\Gamma(u + 1) = u\Gamma u = u! \text{ for } u \text{ a positive integer}$$

since  $\Gamma(1) = 1$ . Because of invertibility, these processes can be equivalently represented by fractional AR processes.

is anti-persistent. When  $d = 0$ , we have white noise  $\varepsilon(t)$  and the ACF decays hyperbolically,  $\gamma(\tau) \rightarrow \sigma_\varepsilon^2 \tau^{-1}$ . When  $d \uparrow \frac{1}{2}$ , the white noise  $\varepsilon(t)$  is fractionally integrated, the ACF decays slower than hyperbolically,  $\gamma(\tau) \rightarrow \sigma_\varepsilon^2$ , a constant, and the series of increments is persistent.

**Definition 8** *Fractional Brownian motion (FBM) is the fractionally differenced series of logarithmic increments  $x(t) = \ln \frac{X(t)}{X(t-1)}$ :*

$$(1-L)^d x(t) = \varepsilon(t), \quad d \in (-0.5, 0.5), \quad \text{with } \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \quad (16)$$

where  $x(t) = \ln X(t) - \ln X(t-1) = (1-L) \ln X(t)$ . Or, equivalently the FBM is the fractionally integrated white noise, since

$$x(t) = (1-L)^{-d} \varepsilon(t), \quad d \in (-0.5, 0.5), \quad \text{with } \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \quad (17)$$

**Example 9** *The standard geometric Brownian motion (GBM) is the special case of fractional Brownian Motion, when  $d = 1$ , so that*

$$\Delta x(t) = (1-L)x(t) = \varepsilon(t) \quad (18)$$

$$\text{or } x(t) = (1-L)^{-1} \varepsilon(t), \quad \text{with } \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \quad (19)$$

Its ACF is

$$\gamma(\tau) \sim \sigma_\varepsilon^2 \tau^{-1} \quad (20)$$

which is proportional to the variance of the i.i.d. innovations  $\varepsilon(t)$ :  $\sigma_\varepsilon^2$ . Thus the GBM is once integrated white noise. The GBM is self-similarly scaling. Its increments are white noise, i.e., they exhibit a flat, constant spectral density:  $P_\varepsilon(\omega) = \sigma_\varepsilon^2$ .

### 4.3 Time-Scale Analysis

The potential occurrence of long-term dependence in FX increments requires an analysis not only of their behavior over time, but also of their frequencies and amplitudes, or analysis at various resolution scales. Time-scale analysis is thus required. Fourier analysis, which assumes stationarity, can't be applied, due to the potential nonstationarity of the FX rates. Windowed Fourier analysis, as computed in Section 2, is imprecise since that "blurs" or "smears" information between windows. Therefore, it can't properly detect discontinuities. and singularities.

To measure long-term dependence and to allow for discontinuities. and singularities in the FX series, Mandelbrot (1972) suggests to use Hurst's rescaled range, or R/S statistic, which Hurst

(1951) developed for his study of the Nile river flows. Hurst quantified and modeled the non-periodic cyclicities in the (sometimes drastic) changes in the Nile's flood behavior to compute the required height for the Aswan dam, a substantial hydrological risk management problem. As we will see, the Hurst statistic leads to the critical Hurst exponent, which can be used to quantitatively characterize long-term dependence behavior.

#### 4.3.1 Hurst Range-Scale Statistic

**Definition 10** Consider a sequence of investment returns  $\{x(t)\}$  and its empirical mean (= first cumulant = first moment)

$$c_1 = m_1 = \frac{1}{T} \sum_{t=1}^T x(t) \quad (21)$$

and its empirical variance (= second cumulant)

$$c_2 = m_2 - m_1^2 = \frac{1}{T} \sum_{t=1}^T (x(t) - m_1)^2 \quad (22)$$

where  $m_2$  is the second moment, then the **Hurst Range-Scale statistic** is

$$RS_H(T) \equiv \frac{1}{c_2^{0.5}} \left[ \text{Max}_{1 \leq \tau \leq T} \sum_{t=1}^{\tau} (x(t) - m_1) - \text{Min}_{1 \leq \tau \leq T} \sum_{t=1}^{\tau} (x(t) - m_1) \right] \geq 0 \quad (23)$$

The first term in brackets is the maximum (over interval  $\tau$ ) of the partial sums of the first  $\tau$  deviations of  $x(t)$  from the mean. Since the sum of all  $\tau$  deviations of  $x(t)$  from their mean is zero, this maximum is always nonnegative. The second term is the minimum (over interval  $\tau$ ) of this same sequence of partial sums; hence it is always non-positive. The difference between these two quantities, called the "range," is thus always nonnegative. This range is scaled by the empirical standard deviation for the whole data set  $\sqrt{c_2}$ .

#### 4.3.2 Hurst Exponent

The Hurst statistic delivers the Hurst exponent as a fractal dimension coefficient, as follows.

**Definition 11** The **Hurst exponent**  $H$  is defined as

$$0 < H = \lim_{\tau \rightarrow \infty} \frac{\ln RS_H(\tau)}{\ln \tau} < 1 \quad (24)$$

For serially, or short term, dependent time series, such as strong-mixing processes,  $H \rightarrow 0.5$  when  $\tau \rightarrow \infty$ , but for globally, or long-term dependent time series  $H \rightarrow 0.5+d$ . In fact, Mandelbrot and Van Ness ' (1968) and Hosking 's (1981) fractionally - differenced time - series satisfies the equality  $H = 0.5 + d$ . Mandelbrot (1972) suggests to plot  $RS_H(\tau)$  against  $\ln \tau$  to compute  $H$  directly from the plot slope.

**Example 12** *The annual water flow of the river Nile in Egypt shows  $H = 0.9$ , a very persistent flow, with occasional large floods. For the rivers Saint Lawrence in Canada, Colorado in the USA, and the Loire in France,  $0.5 < H < 0.9$ . The river Rhine in Germany is exceptional with  $H = 0.5$ , a flow like a random walk, with almost never a large flood (Mandelbrot and Wallis, 1969).*

**Example 13** *Daily observations of the Dow Jones Industrial Average (DJIA), from January 2, 1888 through December 31, 1991, show that the overall Hurst exponent  $H = 0.555$ . In the 1880-1916 period  $H = 0.585$ ; in the 1917-1953 period  $H = 0.565$ ; and in the 1954-1990 period  $H = 0.574$  (Peters, 1994, Chapters 8 and 9). That is more persistent than the  $H = 0.5$  random walk postulated for stock prices by Granger and Morgenstern (1963) and Granger (1966). By measuring stock price increments to be close to Gaussian, Granger and Morgenstern inferred that such increments had thus a typical spectral shape for financial market prices. However, their inference was erroneous, because it was biased by their thinking in term of Gaussian increments  $\varepsilon(t) \sim N(0, \sigma_\varepsilon^2)$ . There is nothing typical about  $H = 0.5$  for financial data series.*

The ACF of the FBM can now be written in terms of the Hurst exponent, since we can now substitute  $d = H - 0.5$ :

$$\gamma(\tau) \sim \sigma_\varepsilon^2 \tau^{2H-2} \text{ as } \tau \rightarrow \infty \quad (25)$$

where  $H \in (0, 1)$ , which shows the FBM to be non-stationary, but time-scaling, since its second moment is a power law of the time lag  $\tau$ . The corresponding average power spectral density of the FBM is:

$$P(\omega) = \sigma_\varepsilon^2 \omega^{-(2H+1)} \quad (26)$$

for frequency  $\omega$ .

The FBM is statistically self-similar in the sense that for any scaling coefficient  $a > 0$ , and with the convention that  $x(0) = 0$ ,

$$x(a\tau) \stackrel{d}{=} a^H x(\tau) \quad (27)$$

where  $\stackrel{d}{=}$  means equality in distribution. This means, in frequency terms, that the average power spectrum of the FBM is frequency-scaling:

$$\begin{aligned}\mathcal{F}[\gamma(a\tau)] &= \frac{1}{|a|^2} P\left(\frac{\omega}{a}\right) \\ &= \frac{\sigma_\varepsilon^2}{|a|^2} \left(\frac{\omega}{a}\right)^{-(2H+1)} \\ &= a^{2H-1} \sigma_\varepsilon^2 \omega^{-(2H+1)}\end{aligned}\tag{28}$$

where  $\mathcal{F}$  is the Fourier Transform of the ACF of the scaled time series.

Thus, any portion of a given FBM can be viewed as a scaled version of a larger part of the same process, both in terms of time and frequency. Consequently, an individual realization of an FBM is a fractal time series and has a unique fractal dimension  $D$  (Mandelbrot, 1966). This fractal dimension and the Hurst exponent  $H$  are related as follows

$$D = 2 - H\tag{29}$$

In summary, the FBM has the features that we seek in an empirical model for FX rates:

- (1) it exhibits nonstationarity, and allows for stationarity as a special case;
- (2) it exhibits self-similarity or time-scale dependence;

Thus, it encompasses integer, serial (ARIMA), and fractional, long-term dependent process models. Table 4 in the Appendix summarizes all mathematical relations between the various Hölder exponents for the increments, *e.g.*, between the difference order  $d$ , the Hurst exponent  $H$  and the Lipschitz regularity exponent  $\alpha_L$ , and some examples of their usefulness.

#### 4.4 Wavelet Multiresolution Analysis

Flandrin (1992) and Mallat (1999) examine the FBM's behavior relative to different observation time and amplitude scales. A second-order moment analysis of the wavelet coefficients of the FBM reveals a stationary structure at each scale and a power-law behavior of the wavelet coefficient's variance, from which the average Lipschitz exponent  $\alpha_L$  of the FBM can be estimated.

Decomposition of a time series into various resolutions through an iteration process is termed a *multiresolution*. At different resolutions, the details of the time series generally characterize different physical structures of the data. Nonparametric wavelet multiresolution is superior to the conventional parametric statistical tests that have been applied to time series analysis, because there is no pre-assumption about the distribution or the processes generating the data, although it allows an easy incorporation of a definite modeling structure, when so required.

Mallat (1989) shows that one can completely decompose any time series  $x(t)$  in terms of approximations, provided by so-called scaling functions, and details, provided by wavelets. The approximations are the high-scale, low-frequency components of the time series. The details are the low-scale, high-frequency components. He formulates these concepts of scale and resolution into mathematical requirements for multiresolution analysis (MRA), by requiring a nesting of spanned spaces  $V_j$  of different levels of resolution, as follows:

$$\mathbf{V}_{j+1} \subset \mathbf{V}_j \text{ for all } j \in \mathbb{Z} \quad (30)$$

with

$$\mathbf{V}_\infty = \{0\} \text{ and } \mathbf{V}_{-\infty} = L^2 \quad (31)$$

Thus the linear space that contains low resolution will also contain the linear spaces of higher resolutions. This means that at a zero resolution, the only finite energy time series is 0, while at the infinite resolution, all finite energy time series are *completely* reproduced in a quadratic measure sense.<sup>18</sup>

This decomposition process can be iterated, with successive approximations being decomposed in turn, so that one time series  $x(t)$  is broken down in many lower-resolution components. Since the analysis process is iterative, in theory it can be continued indefinitely. In reality, the decomposition can proceed only until the individual details consist of a single observation. In our case, that is a one-minute FX increment.

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<sup>18</sup> That is completely and not approximately. No energy = information is discarded! This is important for accurate forecasting (Aussem et al., 1998).

The following discussion of Mallat's MRA is indebted to Jaweth and Sweldens (1994), Burrus, Gopinath and Guo (1998), Frazier (1999), and Mallat (1999). Mallat (1988) proves that, using a combination of these scaling functions  $\varphi_n(t)$  and wavelets  $\psi_{j,n}(t)$ , any time series can be represented by the following decomposition equation.

**Definition 14** *A Wavelet Multiresolution Analysis (MRA) of any time series  $x(t)$  is provided by:*

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \varphi_n(t) + \sum_{j=0}^{+\infty} \sum_{n=-\infty}^{+\infty} d_{j,n} \psi_{j,n}(t) \quad (32)$$

where the approximation is provided by the one-dimensional linear combination of the scaling functions

$$\sum_{n=-\infty}^{+\infty} c_n \varphi_n(t) = \sum_{n=-\infty}^{+\infty} c_n \varphi(t-n) \quad (33)$$

and the details by the two-dimensional linear combination of the dyadic wavelets

$$\sum_{j=0}^{+\infty} \sum_{n=-\infty}^{+\infty} d_{j,n} \psi_{j,n}(t) = \sum_{j=0}^{+\infty} \sum_{n=-\infty}^{+\infty} d_{j,n} \psi(2^{-j}t-n) \quad (34)$$

The discrete scaling coefficients are computed by the correlation

$$c_n = \langle x(t) \phi_n(t) \rangle = \int_{-\infty}^{+\infty} x(t) \phi_n(t) dt, \text{ with } n \in \mathbb{Z} \quad (35)$$

The discrete wavelet coefficients are computed by the correlation

$$d_{j,n} = \langle x(t) \psi_{j,n}(t) \rangle = \int_{-\infty}^{+\infty} x(t) \psi_{j,n}(t) dt, \text{ with } j, n \in \mathbb{Z} \quad (36)$$

**Remark 15** *Thus the structure of Mallat's MRA is isomorph to that of a discrete form of Brownian motion, except that the resolution of the details, or irregularity component, has two dimensions, time and scale, instead of one, time.*

Strang (1989) shows that Mallat's definition of an MRA implies that the scaling function  $\varphi(t)$  can be expressed in terms of an expansion, *i.e.*, a weighted sum of shifted  $\varphi(2t)$  as follows.

**Definition 16** *The MRA (dilation, or scaling) equation is*

$$\varphi(t) = \sum_{n=-\infty}^{+\infty} h(n) \sqrt{2} \varphi(2t-n), \text{ for any } n \in \mathbb{Z} \quad (37)$$

where the coefficients  $h(n)$  are real or complex numbers, called the scaling function coefficients (or the scaling filter or the scaling vector), and the scaling factor  $1/\sqrt{2}$  maintains the norm of the scaling function.

**Remark 17** An equivalent way to present the same MRA equation is

$$\frac{1}{\sqrt{2}}\varphi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h(n)\varphi(t-n), \text{ for any } n \in \mathbb{Z} \quad (38)$$

It's Fourier Transform is

$$\Phi(\omega) = \frac{1}{\sqrt{2}}\mathbf{H}\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \quad (39)$$

where  $\mathbf{H}(\omega)$  is the transfer function, i.e., the Fourier Transform of  $h(n)$ .

**Definition 18** The **MRA equation for wavelets** is the weighted sum of shifted scaling functions

$$\psi(t) = \sum_{n=-\infty}^{+\infty} h_1(n)\sqrt{2}\phi(2t-n), \quad n \in \mathbb{Z} \quad (40)$$

for some set of wavelet generation coefficients  $h_1(n)$ , since the wavelets reside in the space spanned by the next narrower scaling function,  $\mathbf{W}_0 \subset \mathbf{V}_1$ .

**Remark 19** This MRA equation for wavelets can equivalently be presented as

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h_1(n)\phi(t-n), \quad n \in \mathbb{Z} \quad (41)$$

Its Fourier transform is

$$\Psi(\omega) = \frac{1}{\sqrt{2}}\mathbf{H}_1\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \quad (42)$$

The specific MRA equations for scaling functions and wavelets used in this paper are the ones for the very simple Haar scaling function and wavelet. These Haar functions are appropriate for the regularized FX rate increments. The Haar MRAs with resolution scale up to  $j = 8$ , gave the best synthesized series and therefore was chosen to plot the Wavelet-based scalograms. This implies that we use resolutions up to scale level  $j = 2^3 = 8$ , so that we capture as largest features of the FX increments:  $2^8 = 256$  minutes = 4.5 hour = ca. half-a-day trading intervals.

**Definition 20** The **Haar scaling function** is

$$\varphi(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \quad (43)$$

and the Haar MRA equation for the scaling function is

$$\varphi(t) = \varphi(2t) + \varphi(2t-1) \quad (44)$$

with scaling function generation coefficients  $h(0) = h(1) = \frac{1}{\sqrt{2}} = 0.70711$  (rounded to five digits).

**Definition 21** The *Haar wavelet* is defined by

$$\psi_{j,n}(t) \begin{cases} = +1 & \text{if } 0 \leq t < 0.5 \\ = -1 & \text{if } 0.5 \leq t < 1 \\ = 0 & \text{otherwise} \end{cases} \quad (45)$$

The Haar MRA equation for wavelets is

$$\psi(t) = \phi(2t) - \phi(2t - 1) \quad (46)$$

with wavelet generation coefficients  $h_1(0) = -h_1(1) = \frac{1}{\sqrt{2}} = 0.70711$ .

Haar wavelets offer symmetry, orthogonality and compact support. This box function, developed in by Haar (1910), is simple to understand. The Haar basis takes advantage of the self-similarity of the FBM, such that it reduces the bias of the variance progression. Setting the wavelet coefficients to the finest scale, they follow a nice variance progression per scale, and they decay much faster than the discrete fractional Gaussian noise.

#### 4.4.1 Scalograms

The wavelet analogy of the spectrogram is the scalogram, which is simply the time-scale presentation of the squares of the discrete wavelet transform  $d_{j,n}$ .

**Definition 22** A (discrete) *scalogram* is defined by

$$P_W(j, n) = |d_{j,n}|^2 \text{ for all available } j \text{ and } n \quad (47)$$

The scalogram basically represents the energy of the integrated series in the wavelet coefficients. Wavelet coefficients that are below a given threshold value,  $|d_{j,n}| < \epsilon$ , are eliminated by denoising. This paper applies the universal thresholding method, where  $\epsilon = \sqrt{2 \log n}$ . The choice of filter size and the number of levels are important in denoising.

The scalograms of the increments are computed for the last week of June (24th to 30th June, from Tuesday to Monday) and for the first week of July (1st to 7th July which is from Tuesday to Monday = 10,080 observations), to capture the FX data characteristics before and after the currency break on July 2, 1997. The striking visual result is that the scalogram in Fig. 4 ex-

hibits singularity spectra, as the increments of the step functions are singular.<sup>19</sup> Fig. 4 shows original white noise (Wiener increments) in its left panel and denoised white noise in its right panel. Denoising allows better visualization of coefficients, as the remaining coefficients are more pronounced in the denoised scalograms. Noise at the highest frequency (*i.e.*, at the smallest scale) is removed in the denoised scalograms.

[FIGURE 4 about here]

From the scalogram of the DEM increments in June 1997 in Fig. 5, it is again evident that the FX increments are quite different from the white noise in Fig. 4, which has the same variance as the DEM increments. When there is no trading activity, the scalogram displays that with red. For example at the right side of the panels the large red region indicates the no trading weekend (Saturday and Sunday). The left panel of Fig. 5 shows the original DEM increments, while the right panel shows the denoised DEM increments. Notice how denoising increases the resolution of the singularities. The DEM singularities show irregular time spacing - they show periods of condensation and of rarefaction due to rapid trading and (very) slow trading periods. This is not related to sharply defined trading periods, since the DEM is clearly traded 24 hours, five days a week. This striking similarity of the times of trading in the scalograms is observed for all currencies, a reflection of the institutional interconnections of the global FX markets.<sup>20</sup>

[FIGURE 5 about here]

But self-similarity of the FX increments is evident in the scalogram from the fact that the energy levels appear to be almost the same at all resolution scales, *i.e.*, at all trading frequencies for the same singularities.<sup>21</sup> Thus a scalogram visualizes simultaneously timing information and scale (frequency) information, in a similar, but much more precise fashion than a spectrogram.

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<sup>19</sup> The color coding of the scalograms is exactly opposite that of the spectrograms in Section 3, and so is the resolution scaling compared with the frequencies. Red in the scalograms indicates no energy and blue indicates high energy, while yellow and green indicate moderate energy. The small resolution scales = high frequencies are at the bottom, while the large resolution scales = low frequencies are at the bottom of the scalograms. The smallest scale at the bottom represents the individual observations of the minute-by-minute FX rates.

<sup>20</sup> All scalograms are available upon request.

<sup>21</sup> This issue will become clear in Section 5, when we compute the Hurst exponent values for 8 scale levels.

This self-similarity is measured by the homogeneous Hurst exponent in Section 5.

The scalograms for the DEM increments for July 1997 in Fig. 6 (original and denoised) differ in intensity from those in June 1997 in Fig. 5, indicating a higher trading intensity immediately after the break in the Thai baht on July 2, 1997 in Fig. 6, than immediately before the break in Fig. 5.

[FIGURE 6 about here]

This raises the important question: is the fractality of FX series induced by the trading times (day and night sequence) or by the valuation process and can they be analytically separated? The pricing of FX is usually executed in the form of standardized, fixed-sized ticks, say, of  $1/8 = 0.125$ . However, the trading intervals vary and can be a fraction of whatever we choose as unit of measurement of time (*e.g.*, second, minute, hour, day, week, month or year). Thus the fractality of the FX time series must reside in the varying time intervals between the trades and not in the pricing of the trades. Consequently, when the Hurst exponent measures persistence (*Cf.* Section 4.3), the fractality is dependent on its time denominator  $\ln \tau$ , than in its range numerator  $\ln RS_H(\tau)$ , which is dependent on the time horizon  $\tau$ .

The scalograms of most Asian currencies are similar, although with noticeably lower intensity of activity and with much less trading activity. There are large regions of red = no trading activity. Whatever trading activity there is in Asia (with the exception of the JPY, and, perhaps, the SGD, it is less intense and spread much thinner over time and less representable by dense Wiener increments than the anchor currencies JPY and DEM.

The denoised scalogram of Thai baht displays the sharp break on July 2, 1997 extremely well very well, as seen in Fig.7, which is a 3-dimensional representation of the absolute value of the wavelet coefficients, *i.e.*, of the scalogram of the first FX trading week in July 1997.<sup>22</sup> The discontinuity in the Thai baht's value is clearly represented by the energy spike of the wavelet

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<sup>22</sup> Visualization programs are difficult to make consistent. In Fig. 7 blue indicates here lower energy and red high energy, with yellow and green moderate levels of energy.

coefficients.

[FIGURE 7 about here]

#### 4.4.2 Computing Hurst Exponents From Haar MRA

The uniform Lipschitz regularity exponent  $\alpha_L$ , in particular, the homogeneous Hurst exponent for the FBM, can be computed from the FX increments, using the wavelet detail coefficients  $d_{j,n}$  from Mallat's MRA, *i.e.*, from the scalogram (*Cf.* Wornell, 1993).

These Haar wavelet coefficients of the FBM have the following four properties, as proved by Flandrin (1992):

- (1) The wavelet coefficients are stationary in distribution:

$$d_{j,n} \stackrel{d}{=} d_{j,0} \text{ for all } n \quad (48)$$

- (2) The wavelet coefficients are Gaussian in the limit:

$$d_{j,n} \sim N(0, \text{Var}(d_{j,n})) \quad (49)$$

- (3) The wavelet coefficients are almost uncorrelated:

$$E \{d_{i,n} d_{j,m}\} \simeq |2^{-i}n - 2^{-j}m|^{2(H-R)} \quad (50)$$

- (4) The wavelet coefficients scale:

$$d_{j,n} \stackrel{d}{=} 2^{jH} d_{0,n} \quad (51)$$

Flandrin (1992) and Kaplan and Kuo (1993) proved that the variance of these wavelet coefficients  $d_{j,n}$ , computed as averages over the time shifts  $n$  at each resolution level  $j$ , of the FBM is represented by the power law:

$$\text{Var} \{d_{j,n}\} = E \left\{ |d_{j,n}|^2 \right\} = \frac{\sigma_\varepsilon^2}{2} V_\psi(H) (2^j)^{2H+1} \quad (52)$$

where the constant  $V_\psi(H)$ , which depends on both the ACF of the chosen wavelet  $\psi$ , and on the Hurst exponent  $H$ , is defined by:

$$V_\psi(H) = - \int_{-\infty}^{+\infty} \gamma_\psi(\tau) |\tau|^{2H} d\tau \quad (53)$$

with the ACF  $\gamma_\psi(\tau)$  of the wavelets  $\psi(t)$ :

$$\gamma_\psi(\tau) = \int_{-\infty}^{+\infty} \psi(t)\psi(t-\tau)dt \quad (54)$$

Thus, by taking the dyadic logarithm of  $Var\{d_{j,n}\}$ , we find the linear relationship from which we can compute  $H$ :

$$\log_2 [Var\{d_{j,n}\}] = (2H + 1)j + \log_2 \left[ \frac{\sigma_\varepsilon^2}{2} V_\psi(H) \right] \quad (55)$$

Since the second term is a constant, one can plot  $\log_2 [Var\{d_{j,n}\}]$  against the scale coefficient  $j$  to find the slope  $(2H + 1)$  and thus the Hurst exponent  $H$ .

In the wavelet literature, usually the  $2^{-5}$  scale (*i.e.*,  $j = 1, 2, \dots, 5$ ) is deemed as the ideal scale to compute the Hurst exponent from Haar wavelets. However, this would not allow us to capture the values, which could indicate multi-fractality. Hence  $H$ -values were computed from resolution scales  $2^{-2}$  to  $2^{-8}$  (*i.e.*,  $j = 2, 3, \dots, 8$ ). Each level of scaling leads to a decrease in sampling by a factor of 2. The reason for going up to  $2^{-8}$  is to capture the actual sampling of the very thinly traded FX series in the Asian markets. The first scale  $j = 1 = 2$  minutes is considered noise for this computation.

Figs. 8 and 9 provides examples of the computed Hurst exponents for the Deutschemark and the Japanese Yen, respectively, for the months May - August 1997, based on resolution scales  $2^{-2}$  to  $2^{-8}$ .

[FIGURE 8 about here]

[FIGURE 9 about here]

Notice that the Hurst exponents of all resolution scales are virtually the same, corroborating the fractality, or self-similarity, of the FX increments. There are small variations from month to month. In June and July the FX increments were slightly less persistent than in May or August, The maximum monthly variation is  $\pm(0.05)$ . This was also true for the other Asian FX increments. The Hurst exponents are non-homogeneous or multifractal in the  $j = 2 - 5 = 4 - 32$  minute resolution, but the mean  $H$  is homogeneous in the  $j = 5 - 8 = 32$  minute to 4.3 hour

resolution. Only computation for all the months of 1997 can show the complete range in variation of persistence throughout the year 1997.

## 5 ANTI-PERSISTENT FX RATES

From the dyadic logarithmic plots, we computed the homogeneous Hurst exponents  $H$ , fractal dimensions  $D = 2 - H$ , and Zolotarev stability exponent  $\alpha_Z = 1/H$  for all nine currencies for the four months surrounding the onset of the Asian Financial Crisis on July 2 1997. These analytic results are summarized in Table 3.

[TABLE 3 about here]

Notice the overall anti-persistence of these FX rates. In particular, the DEM and the JPY were consistently anti-persistent with  $0.24 \leq H \leq 0.36$  for all four months. The FX rates return continuously to the point where they came from and behave, in a sense, mean-reverting processes, except that these are long-term dependent series. Such anti-persistent time series can contain regular financial turbulence as a friction-reducing, and thus risk-reducing device.<sup>23</sup>

In contrast, notice that the HKD, the MYR, and the SGD were only mildly anti-persistent with  $0.42 \leq H \leq 0.48$ . The PHP showed more or less geometric Brownian motion with its  $0.43 \leq H \leq 0.52$ . Surprisingly, the TWD was mostly persistent with  $0.49 \leq H \leq 0.67$ , presumably, because that currency's value is administratively controlled and not determined by freely operating market forces. The THB was more strongly anti-persistent in May and June, *i.e.*, in the pre-currency-break months, with  $0.36 \leq H \leq 0.39$ , than in July and August, *i.e.*, in the post-currency-break months, with  $0.43 \leq H \leq 0.47$ , when it showed a clear (nonlinear) trend when it was consistently losing value versus the US dollar. The IDR consisted only of a few discontinuities with  $H = 0.06$  in May (when it did not trade at all!), but exhibited almost Brownian motion in the following three months with  $0.46 \leq H \leq 0.48$ .

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<sup>23</sup> Quantifiable measurement of financial turbulence in speculative markets, in particular in the FX markets is the direction of our current research. Just recently Stoll (2000) proposed financial research of friction in financial markets as a very valuable enterprise, although the analytic methodology he employs in that article is outmoded and proven unscientific.

These empirical results strongly suggest that the Hurst exponents are not really homogeneous, as we maintain hypothetically for our analysis. Inhomogeneous Hurst exponents or pointwise Lipschitz exponents point in the direction of the necessity to measure a multifractality spectrum. This also suggests that even the FBM may not be the perfect model for FX pricing, since the FBM is a model with a homogeneous Hurst exponent, or uniform Lipschitz irregularity exponent  $\alpha_L$ . However, the FBM is already a considerable improvement over the GBM, since it exhibits long term dependence and nonlinear time-scaling, while the GBM does not.

The periods of increases in kurtosis, which we may call condensation periods, as measured by the monthly kurtosis exponent  $\alpha_Z = \frac{1}{\alpha_L} = \frac{1}{H}$  are followed by periods of decreases in kurtosis, or rarefaction periods. It appears that most of the condensation occurred in the month of June, followed by July 1997. In other words there was just more and faster trading of very small amplitude with occasional large price jumps around the hand-over of Hong Kong. Interestingly, already in May the IDR experienced very high kurtosis, *i.e.*, a few sharp trading bursts with either very small or very large increments, already in May, followed by a sharp drop in the kurtosis, *i.e.*, with more evenly dispersed trading activity in smaller valuation steps, in the subsequent months.

We conclude that FX increments are potentially non-homogeneous, multifractal random processes, of which the density distributions change kurtosis over time, and sometimes rather drastically. This corroborates the earlier finding that the third moments of these distributions are nonzero (= non-Gaussian skewness) and the fourth moments non-Gaussian and that both are time-varying. Therefore, complete time-dependent fractality (or singularity) spectra should be computed.

In other words, the results of this exploratory paper results suggest that one must determine the spectrum of pointwise irregularity of the FX increments and then determine when and how kurtosis changes and under the impact of what factors. But that is the subject of another paper.

## 6 CONCLUSIONS

The fractal or self-similar nature of the FX increments, visualized in the scalograms by an even distribution of the pricing energy over the various scales and frequencies, is measured by the Hurst exponent. The empirical findings of this paper support the Fractal Market Hypothesis (FMH) of Mandelbrot (1966, 1971) and Peters (1994), which describes the events in the FX markets better (but still not completely!), than the Efficient Market Hypothesis (EMH) of Fama (1970, 1991) based on martingale theory.

This paper uses various time series processing techniques on high frequency, intra-day, minute-by-minute FX rates of the German Deutschemark and eight Asian currencies to characterize the FX market pricing. Spectral analysis shows that the spectral power of the FX increments resides mostly in the smallest frequencies, i.e. in the fast trading with small price steps, which is not the same as "noise" trading.

The spectrograms of the increments offer a clear visualization of the persistence differences between the various markets. Most Asian FX rates - the Malaysian ringgit, the Philippine pesos, the Thai baht and the Taiwan dollar - display changes in frequency (*i.e.*, non-stationarity) in July 1997, when the Asian Financial Crisis began. The Deutschemark, Japanese Yen and Singapore dollar are stationary, proving that these currencies were not greatly affected by the turbulence in the Thai baht, but continued to trade as before. The spectrograms of the FX increments verify that they are not white noise or Wiener processes, a conclusion that is corroborated by the wavelet based scalograms. Hence the FX rates of the nine currencies do not follow geometric Brownian motions.

The scalograms, which provide both scale and time information, reveal self-similarity of the FX increments at various scales. A closer look at the scalograms of the last week in June and the first week in July suggests that the fractal nature of FX pricing is more induced by the timing of trading activity, than by the actual valuation processes. This particular aspect of the microstructure of

FX markets can reveal more about the FX data generating process. Hence further research is required in that direction, but needs to use unregularized tick-by-tick FX data, instead of our regularized FX rates to avoid aliasing of the results.

This paper uses wavelet multiresolution to compute homogeneous Hurst exponents. The Kaplan Kuo (1993) method, which is a modification of Wornell and Oppenheim's (1992) method, is applied to the FX increments. The graphic fractal dimension  $D$  of all the FX increments, with the exception of the new Taiwan dollar, lies between 1.5 and 2, indicating that the FX increments are anti-persistent. The New Taiwan dollar is exceptionally prone to sharp and completely unexpected discontinuities, induced by administrative control.

The Hurst exponent values of the Deutschemark and the Japanese Yen reveal strong anti-persistence in the  $H = 0.2 - 0.3$  range. This should warn speculators against taking long positions in these anchor currencies. Dynamic valuation models, such as the Black-Scholes equation, which is based on Itô processes, in particular geometric Brownian motion, is likely to result in inaccurate pricing of financial instruments in these anti-persistent markets. However, most Asian FX markets, except the Japan Yen, are less anti-persistent and their Hurst exponents values are closer to 0.5, suggesting that the geometric Brownian motion would provide a correct law of motion. Unfortunately, the FX increments show a much wider dispersion and thus more uncertainty about actual value of their Hurst exponents and thus about their degrees of persistence. That makes valuation in these very unpredictable markets extraordinarily hazardous. A move towards a currency "snake," and ultimately to a currency union, would be very desirable for the ASEAN countries, but is impossible under the current circumstances.

Varying Hurst exponent values across scales and months are an indication of multi-fractality, that is the occurrence of different Lipschitz irregularity at different scales. This added complexity certainly poses a problem in modeling the FX pricing processes. We suspect that it is necessary to gain a better understanding of the non-synchronous timing of FX trading activity to improve the valuation modeling of these markets.

According to the Fractal Market Hypothesis (FMH), the existence of investors with different investment horizons  $\tau$  ensures the continuity of the FX markets. Any instability or discontinuity will be absorbed, once the investors assess the value of the information and its impact on their investment horizon and act accordingly in feedback fashion - hence the observed anti-persistence - to bring about stability.

The driving force behind the Efficient Market Hypothesis (EMH) is that there are many investors with similar objectives and risks. It assumes that all investors are rational and everyone acts on the same information set and the same time horizon  $\tau$ . Thus the EMH is neither able to explain anti-persistence or mean-reverting, nor instabilities or discontinuities. The EMH ignores the importance of trading liquidity, which could actually lead to investors transacting at a price that is different from their assessed fair value. The EMH is limited, especially so for the FX markets, as the movements in the FX rates are not directly tied to economic activity. The FX markets are not used to raise capital, unlike other security markets. FX markets are trading market dominated by arbitragers with different horizons  $\tau$ , actively transacting to take advantage of price discrepancies. For these peculiar reasons the FMH can offer a better explanation of the laws of motion of FX rates than the EMH.

FX markets exhibit nonlinear dynamic structures, high degrees of small amplitude and fast, non-informational, trading and nonperiodic cyclicities. This behavior is possibly induced by frequent international capital flows induced by non-synchronized, but cyclically occurring national business cycles, rapidly changing political regimes and country risk perceptions, unexpected informational shocks to investment opportunities, and, in particular, investment strategies to synthesize and diversify risk claims, using cash swaps between the various national asset markets.

Using of high frequency data might lead to complications. Momentary reactions to news may be too complex to be analyzed by such small FX increments. Most research has used some form of sub-sampling, which may induce artificial dependencies.

Since information trading occurs in lower frequencies, in about the two hour periods according

to Ramsey and Zhang (1997), this is, in principle, encompassed by our largest 4.3 hour resolution scale. Still, high frequency data might not reveal much about this kind of information trading. This shortcoming can easily be overcome by increasing the levels of resolution of the scalograms (but one needs very high resolution monitors, and subsampling windows, to continue to discern any details!).

By making the FX data regular, their distribution is slightly altered or aliased and therefore may not be a good representation of the actual FX processes. Although regular intervals facilitate the wavelet multiresolution, irregular intervals can be used by dictionaries of non-uniform wavelets. Therefore the research should steer in the direction of using the actual irregular ask and bid data to model the actual data generating processes. The use of mid prices may not reflect the different structure in the demand and supply side of each market. We do notice considerable differences between the scalograms of the bid and of the ask FX rates.

## 7 APPENDIX: CRITICAL HÖLDER EXPONENTS

The connection between the various dependence exponents and the currently popular stable distributions is as follows. The dependence exponent  $\lambda$  of the ACF of the long-term dependent time series in section 4.2 equals

$$\lambda = 2d - 1 = \nu - 1 = 2H - 2 = \frac{2}{\alpha_Z} - 2 = 2\alpha_L - 2 \quad (56)$$

where  $d$  is the difference operator (or order) exponent,  $\nu$  the spectral exponent,  $H$  the Hurst exponent,  $\alpha_Z$  the stability exponent of the Zolotarev parametrization of stable distributions, and  $\alpha_L$  the Lipschitz irregularity exponent.<sup>24</sup> For completeness:  $\frac{\lambda}{2}$  is the time-scaling exponent.

The complete spectrum of empirical irregularity in terms of these five critical Hölder exponents is given in Table 4.<sup>25</sup> It summarizes all relationships between the exponents of the increments,

<sup>24</sup> Somewhat confusingly in the literature, the Zolotarev stability alpha  $\alpha_Z = 1/\alpha_L$ , where  $\alpha_L$  is the Lipschitz regularity exponent, or alpha.

<sup>25</sup> The fractal Lipschitz regularity exponents are also called Hölder exponents. Hölder (1859 - 1937) was a German mathematician, who devised treatment of divergent series of arithmetic summations, which led to a regularity exponent now recognised to be similar to Hurst's.

or first differences, of Fractional Brownian Motion (FBM).

[TABLE 4 about here]

For example, for the geometric Brownian Motion increments  $\varepsilon(t)$ , which are white noise or Wiener processes:

$$\lambda = -1, d = 0, v = 0, H = 0.5, \alpha_Z = 2 \quad (57)$$

thus

$$x(t) = (1 - L)^0 \varepsilon(t) = \varepsilon(t) \quad (58)$$

Fractional integration of white noise, when  $d = 0.5$  and  $H \uparrow 1$ , results in pink noise

$$x(t) = (1 - L)^{-0.5} \varepsilon(t) \quad (59)$$

One full integration of white noise, when  $d = 1$ , results in brown noise (= Brownian motion)

$$x(t) = (1 - L)^{-1} \varepsilon(t) \quad (60)$$

In the case of  $0.5 < H < 1$ , the vital property of the fractional Brownian motion (FBM) is that the persistence of its increments extends forever: *it never dies out* and gives rise to *non-periodic cyclicities*. The strength of such persistence is measured by the Hurst exponent.

The case where  $0.5 < d < 1.5$ , or, equivalently,  $1 < v < 3$  has been called the *infrared catastrophe* (Wornell & Oppenheim, 1992). It could occur in the securities markets, but is unlikely to occur in the FX markets, which are closer to blue or chaotic noise, which is extremely stably distributed. More fractional integration, for example  $d = 2$ , results in heavily persistent, or black noise

$$x(t) = (1 - L)^{-2} \varepsilon(t) \quad (61)$$

As Schroeder (1991, p. 122) comments: "Black-noise phenomena govern natural and unnatural catastrophes, like floods, droughts, bear markets, and various outrageous outages, such as those

of electrical power. Because of their black spectra, such disasters often come in clusters.” If anything, it can be caused by administrative interventions in the markets or by illiquidity.

The FBM increments with  $0 < H < 0.5$  are antipersistent noise, hence they diffuse faster than the Brownian increments. Consequently, the FBM returns continuously to the point it came from and behaves more like a mean-reverting process.

Notably this means that white noise increments  $\varepsilon(t)$  are rather exceptional. They exhibit the same stability,  $\alpha_Z = 2$ , and (in-)dependence,  $H = 0.5$ , as Gaussian random variables (but do not necessarily have to be Gaussian!) Furthermore, their ACF drops of geometrically with  $\lambda = -1$ .

It is very important to understand that the Hurst exponent is a rather limited measure of randomness and distributional stability with a very limited domain, and that the  $\alpha_Z$ -stability exponent, respectively the  $v$ -spectral exponent, have much more extensive domains. There exist empirical ultra-stable distributions (not yet parametrized!) in the domain  $2 \leq \alpha_Z < \infty$ , since we find *in extremo*  $\alpha_Z \uparrow \infty$  when  $H \downarrow 0$  (and  $d \uparrow 0.5$ ), which is complete stability. These are the distributions of singularities, or singularity spectra, which can still be measured by the stability exponent  $\alpha_Z$ .

In addition, there are now theoretically defined, parametrized stable distributions where  $0 < \alpha_Z < 1$ , which cannot be directly measured by the Hurst  $H$ -exponent, but can be measured by  $\alpha_Z$ , if we can compute  $\alpha_Z$  in some other fashion. These are the theoretical ultra-unstable distributions. However, empirically there appears to be a turbulence barrier at  $\alpha_Z = 2/5$ . In other words, there appear not to exist any empirical  $0 < \alpha_Z < 2/5$ , even though there are theoretical Zolotarev-parametrized distributions defined for such  $\alpha_Z$  values.

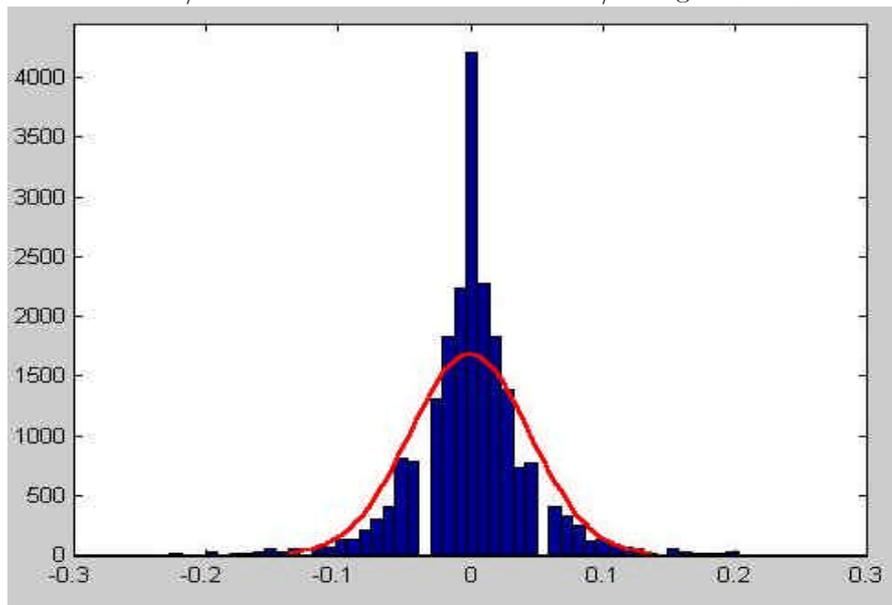
Of course, one can still use the Hurst exponent for measuring infrared and black catastrophes, by measuring it after proper integer-differentiation. For example, when we hypothesize that  $x(t)$  is pure black noise and has a spectral exponent  $v = 4$ , then differentiation two full times ( $d = 2$ ) should theoretically result in white noise with a flat spectrum,  $v = 0$ , so that  $H = 0.5$ . However, when we then empirically measure, for example,  $H = 0.2 \rightarrow v = -0.6$ , then the original series

must have a spectral coefficient of  $v = -0.6 + 4 = 3.4$  and not 4.

## 8 FIGURES

### 8.1 Figure 1

Documents/Asian FX Wavelet Multiresolution/HistogramJPY0.wmf

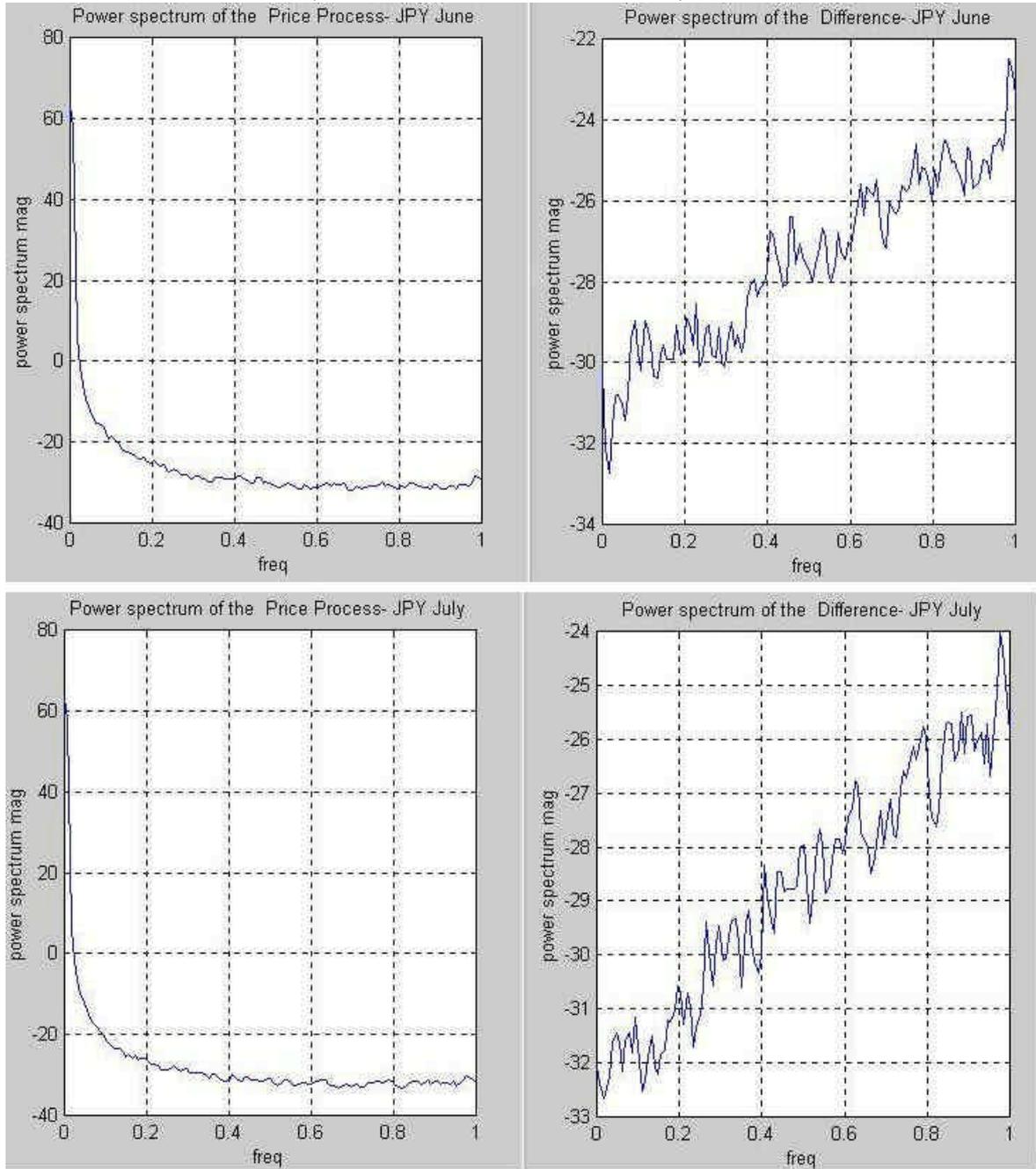


JPY June 1997

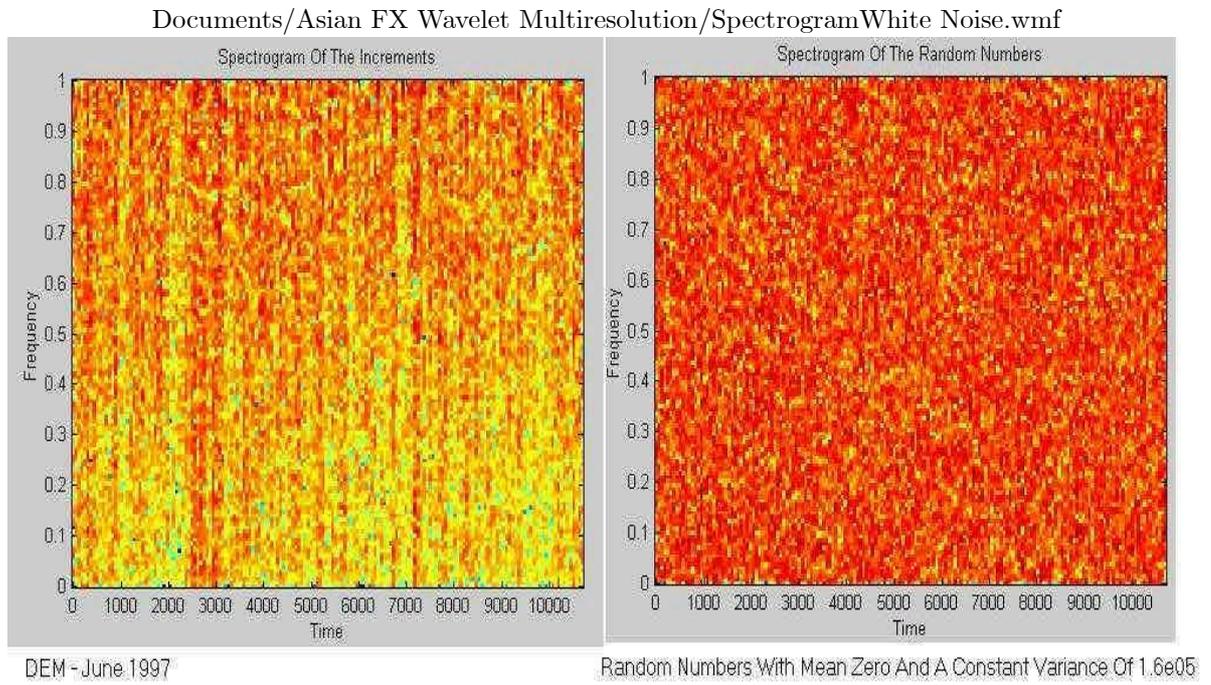
Histogram of Japanese Yen for the Month of June 1997

## 8.2 Figure 2

Documents/Asian FX Wavelet Multiresolution/jppower.jpg

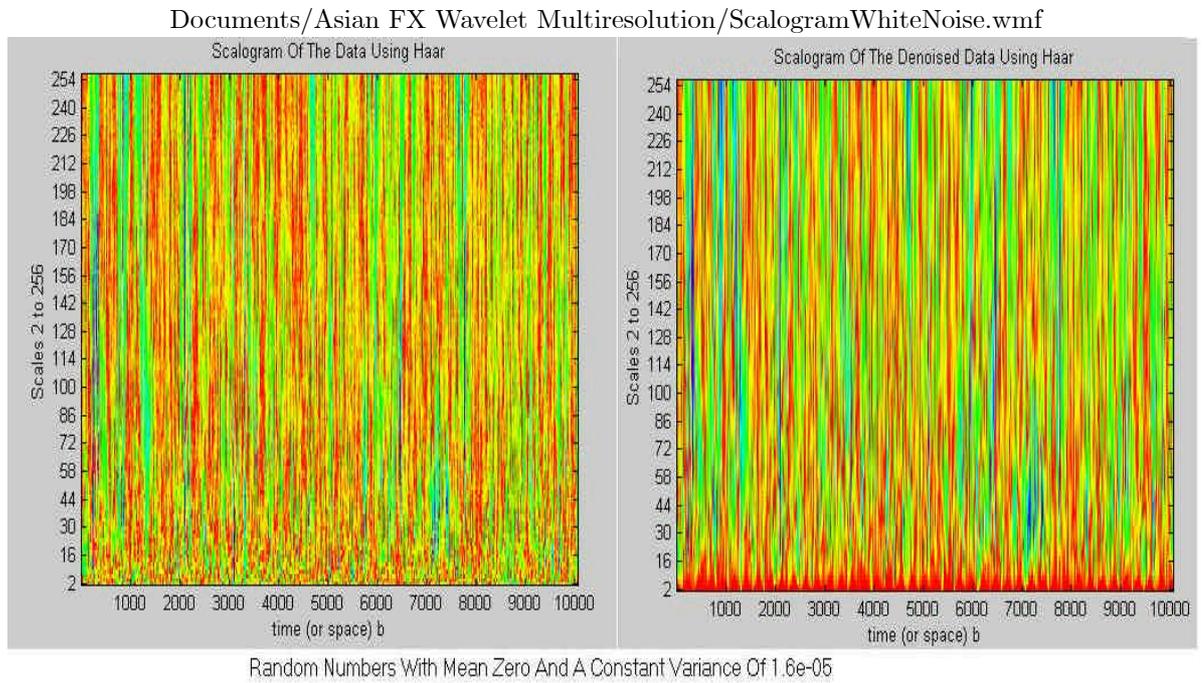


### 8.3 Figure 3



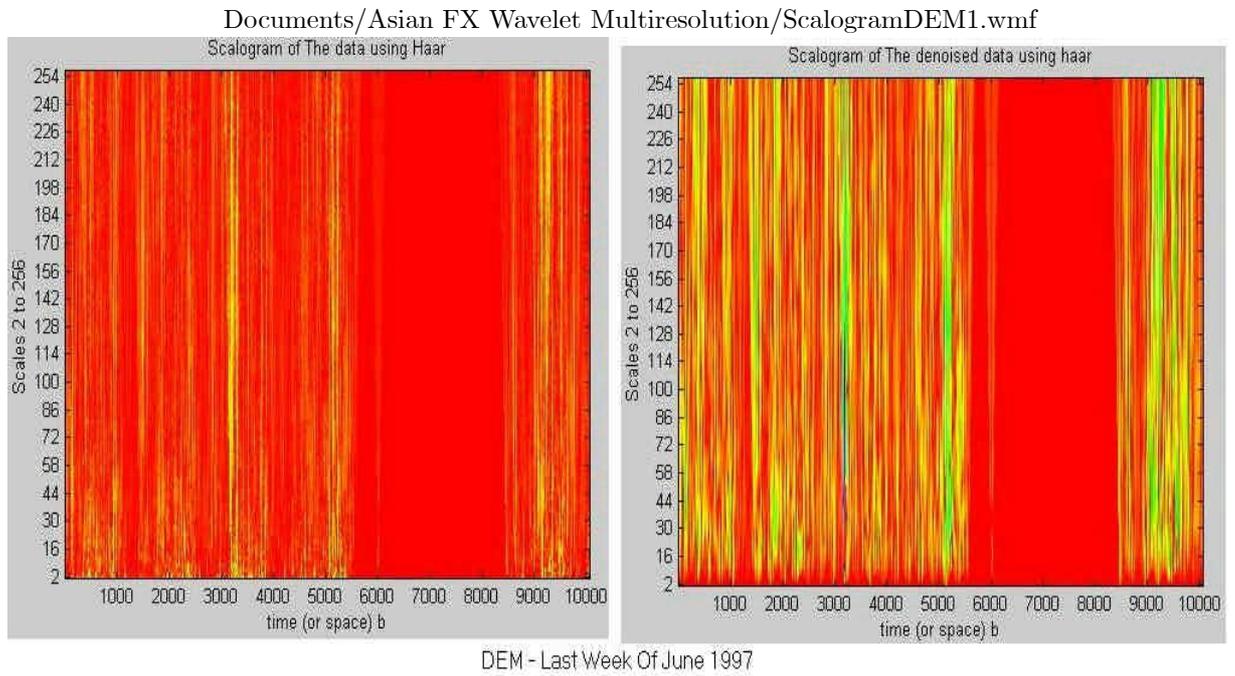
Spectrogram of DEM Increments and White Noise With Same Variance

### 8.4 Figure 4



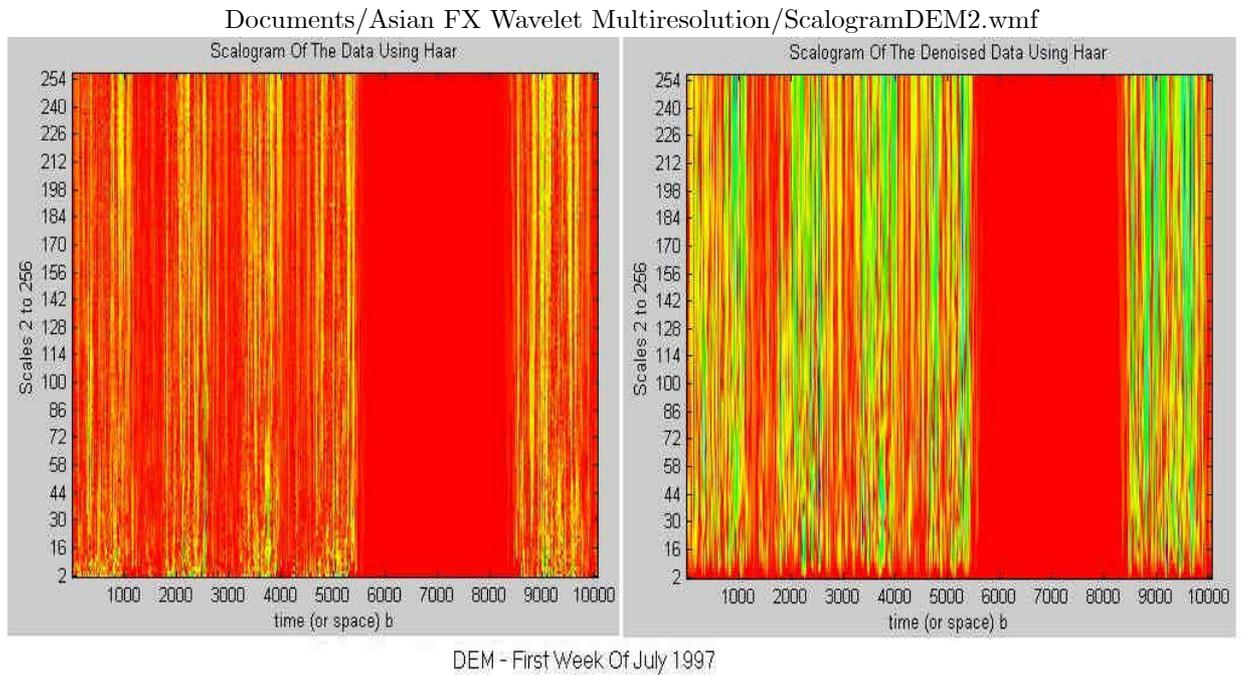
Original And Denoised Scalogram Of White Noise

### 8.5 Figure 5



Original And Denoised Scalogram Of DEM For The Last Week Of June 1997

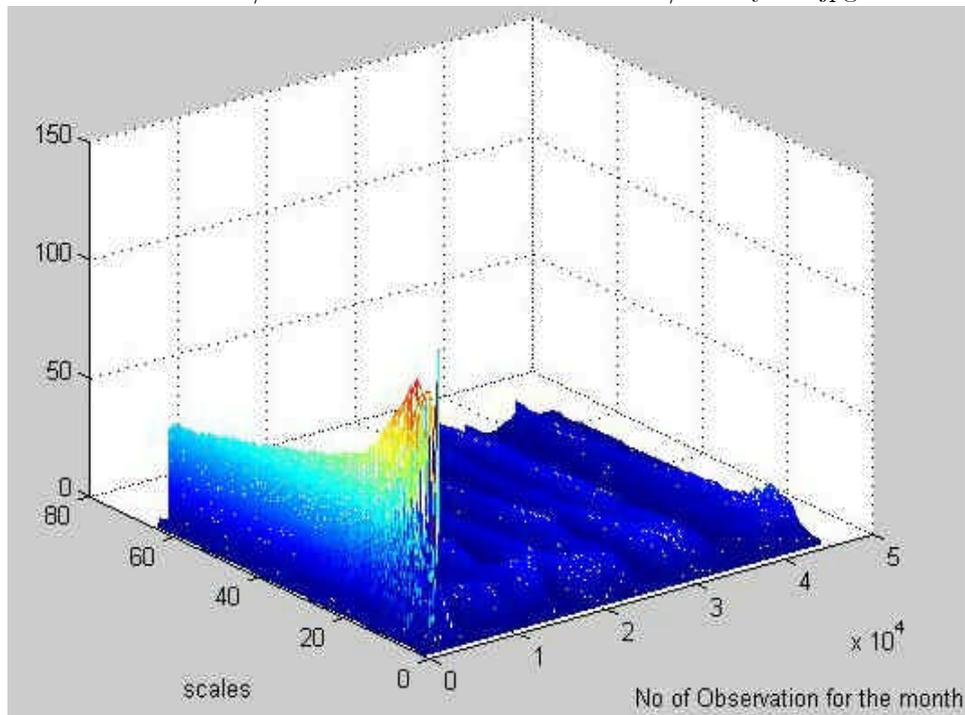
8.6 Figure 6



Original And Denoised Scalogram Of DEM For The First Week Of July 1997

8.7 Figure 7

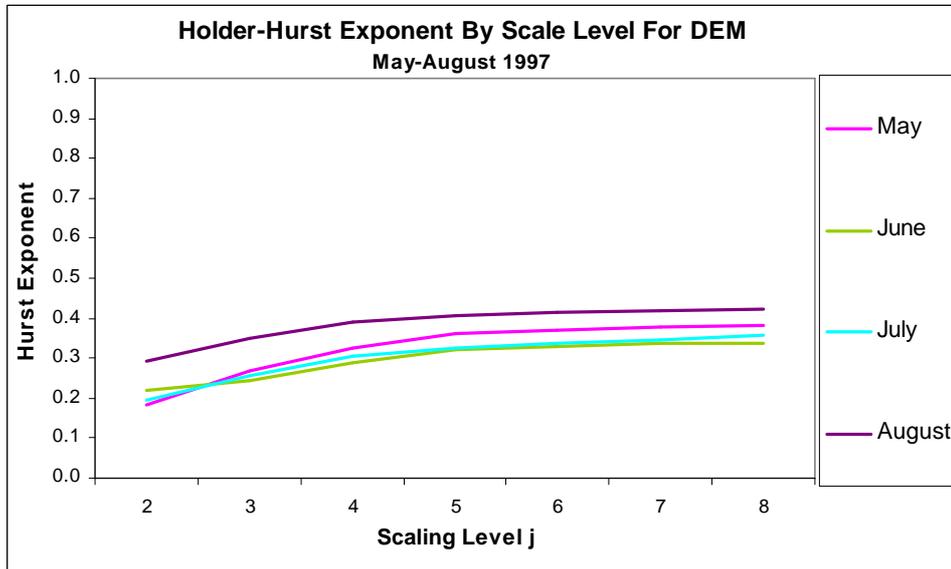
Documents/Asian FX Wavelet Multiresolution/ThJulyDen.jpg



3D View of the scalogram of the Thai baht increments of July 1997

### 8.8 Figure 8

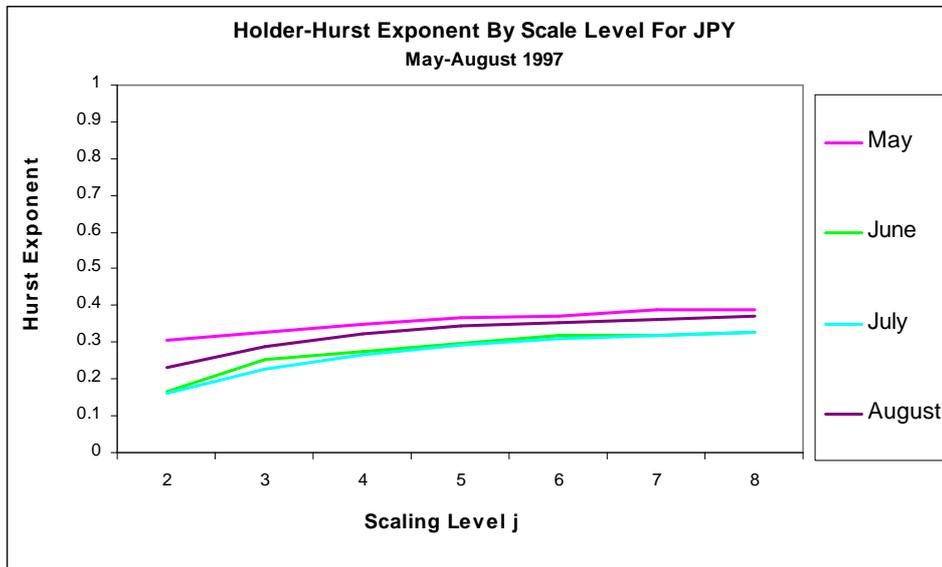
Documents/Asian FX Wavelet Multiresolution/DyadicPlotDEM.wmf



Dyadic Logarithmic Plot of the DEM Hurst Exponents

### 8.9 Figure 9

Documents/Asian FX Wavelet Multiresolution/DyadicPlotJPY.wmf



Dyadic Logarithmic Plot of the JPY Hurst Exponents

## 9 TABLES

### 9.1 Table 1

Documents/Asian FX Wavelet Multiresolution/StatisticsIncrements.wmf

**TABLE1: DISTRIBUTIONAL STATISTICS FOR THE NINE FX RATE INCREMENTS,  
MAY – AUGUST 1997.**

<b>DEM</b>	May	June	July	August
mean	-1E-06	1.72E-06	4.01E-06	-1.4E-06
std	0.000443	0.000397	0.000451	0.000519
kurt	7.488048	3.587991	5.938828	4.633
skew	-0.28438	0.066015	0.073748	-0.14794

<b>PHP</b>	May	June	July	August
mean	0	-3.9E-05	0.030856	0.021264
std	0.002919	0.004808	0.588818	0.17746
kurt	0.381158	2.189639	16.36673	8.58714
skew	-0.14157	-0.09223	2.701964	-0.9213

<b>JPY</b>	May	June	July	August
mean	-0.00055	-5.6E-05	0.000165	0.000106
std	0.052907	0.04544	0.038622	0.044481
kurt	12.30398	11.66099	5.131492	9.739176
skew	-0.44484	-0.01558	-0.03453	-0.08742

<b>SGD</b>	May	June	July	August
mean	-4.3E-06	0	8.16E-06	8.97E-06
std	0.000354	0.000322	0.000472	0.0008
kurt	<b>26.00865</b>	6.769136	10.84123	14.74747
skew	-0.96307	-0.32234	-0.05337	-0.59471

<b>HKD</b>	May	June	July	August
mean	1.99E-06	-2.1E-06	-3.1E-06	3.35E-06
std	0.000365	0.000394	0.000385	0.000512
kurt	2.693181	2.92056	4.024717	18.42473
skew	0.401515	-0.11205	-0.17704	-0.20931

<b>THB</b>	May	June	July	August
mean	-0.00091	-0.00035	0.004674	0.001396
std	0.034863	0.113586	0.171863	0.141878
kurt	<b>43.33442</b>	<b>89.80316</b>	<b>137.9081</b>	13.66864
skew	-1.86699	0.700684	6.349645	-0.39254

<b>IDR</b>	May	June	July	August
mean	0.000655	0.00032	0.069487	0.094868
std	0.910563	0.30992	2.9969	6.835453
kurt	<b>265.8008</b>	5.483776	8.916913	11.39796
skew	-3.39791	-0.18143	0.235687	0.384675

<b>TWD</b>	May	June	July	August
mean	0.000157	-3.8E-05	0.000708	3.78E-05
std	0.005855	0.003579	0.010136	0.017432
kurt	7.61613	4.466143	<b>42.21601</b>	<b>66.90014</b>
skew	0.578374	0.152799	1.831768	-4.80517

<b>MYR</b>	May	June	July	August
mean	5.31E-07	2.96E-06	2.4E-05	6.51E-05
std	0.000634	0.00044	0.001286	0.002338
kurt	16.41839	5.816113	<b>62.33748</b>	29.10995
skew	-0.83357	-0.34258	2.513235	1.494065

## 9.2 Table 2

Documents/Asian FX Wavelet Multiresolution/Gaussianity.wmf

TABLE 2: GAUSSIANTY AND LINEARITY TESTS					
Statistic For Gaussianity			Statistics for Linearity		
	PFA		B-estimated	B-Theoretical	
<b>DEM</b>					
June	1	Gaussian	0.1867	2.3738	Linear
July	1	Gaussian	0.4619	2.5449	Linear
<b>JPY</b>					
June	0	Non-Gaussian	1.8697	4.2577	Linear
July	1	Gaussian	0.0806	2.2431	Linear
<b>HKD</b>					
June	0	Non-Gaussian	5.8338	5.8308	Linear
July	0	Non-Gaussian	4.7732	5.5955	Linear
<b>IDR</b>					
June	0	Non-Gaussian	7.2458	6.8485	Linear
July	0	Non-Gaussian	14.0718	8.9655	Non-Linear
<b>MYR</b>					
June	0	Non-Gaussian	4.0577	5.1577	Linear
July	0	Non-Gaussian	32.2052	14.9589	Non-Linear
<b>PHP*</b>					
June	0	Non-Gaussian	4.7352	5.3152	Linear
July	0	Non-Gaussian	70.4668	14.8064	Non-Linear
<b>SGD</b>					
June	0	Non-Gaussian	2.9047	5.4764	Linear
July	1	Gaussian	1.0734	3.1847	Linear
<b>THB</b>					
June	0	Non-Gaussian	173.9235	34.4184	Non-Linear
July	0	Non-Gaussian	299.7581	40.3008	Non-Linear
<b>TWD</b>					
June	0	Non-Gaussian	3.7959	4.4522	Linear
July	0	Non-Gaussian	170.7834	30.1341	Non-Linear

- The length of data is below 128 and therefore the results pertaining to PHP are not reliable
- PFA = Probability of False Alarm
- R = Bicoherence

### 9.3 Table 3

Documents/Asian FX Wavelet Multiresolution/HurstExponents.wmf  
**TABLE 3: VALUES OF HOMOGENEOUS HURST EXPONENTS  
 FOR NINE CURRENCIES IN MAY- AUGUST 1997**

	MAY			JUNE			JULY			AUGUST		
	H	D	$a_z$	H	D	$a_z$	H	D	$a_z$	H	D	$a_z$
DEM	0.28	1.72	3.52	0.27	1.73	3.72	0.27	1.73	3.71	0.36	1.64	2.78
JPY	0.34	1.66	2.96	0.25	1.75	4.03	0.24	1.76	4.23	0.30	1.70	3.37
HKD	0.45	1.55	2.21	0.46	1.54	2.19	0.47	1.53	2.11	0.46	1.54	2.16
IDR	0.06	1.94	16.25	0.48	1.52	2.10	0.46	1.54	2.17	0.47	1.53	2.15
MYR	0.45	1.55	2.20	0.42	1.58	2.38	0.46	1.54	2.19	0.48	1.52	2.07
PHP	0.52	1.48	1.93	0.43	1.57	2.32	0.51	1.49	1.97	0.49	1.51	2.03
SGD	0.44	1.56	2.30	0.46	1.54	2.17	0.45	1.55	2.21	0.42	1.58	2.37
THB	0.36	1.64	2.77	0.39	1.61	2.56	0.47	1.53	2.13	0.43	1.57	2.34
TWD	0.55	1.45	1.81	0.55	1.45	1.82	0.67	1.33	1.49	0.49	1.51	2.06

Note: The fractal dimension  $D = 2 - H$ ; Zolotarev's stability exponent  $\alpha_Z = 1/\alpha_L$ , the inverse of the Lipschitz irregularity exponent  $\alpha_L$ , as measured by the Hurst exponent H.

### 9.4 Table 4

TABLE 4	HÖLDER	EXPONENTS	MEASURE	IRREGU	LARITY
EXPONENTS:	DEPENDENCE	DIFFERENCE	SPECTRAL	HURST	STABILITY
COLOR:	$\lambda$	$d$	$v$	$H$	$\alpha_Z$
Blue noise	$\lambda \downarrow -2$	$d = -0.5$	$v = -1$	$H \downarrow 0$	$\alpha_Z \uparrow \infty$
Anti-persistent noise	$-2 < \lambda < -1$	$-0.5 < d < 0$	$-1 < v < 0$	$0 < H < 0.5$	$2 < \alpha_Z < \infty$
White noise	$\lambda = -1$	$d = 0$	$v = 0$	$H = 0.5$	$\alpha_Z = 2$
Persistent noise	$0 < \lambda < -1$	$0 < d < 0.5$	$0 < v < 1$	$0.5 < H < 1$	$1 < \alpha_Z < 2$
Pink noise	$\lambda \uparrow 0$	$d = 0.5$	$v = 1$	$H \uparrow 1$	$\alpha_Z = 1$
Brown noise	NA	$d = 1$	$v = 2$	NA	$\alpha_Z = 2/3$
Black noise	NA	$1 \leq d \leq 2$	$2 < v \leq 4$	NA	$2/5 \leq \alpha_Z < 2/3$

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